PLANCKS Austria 2025

Graz, 5th - 6th April

Team Name

Title	**Points**	
Problem Quartet	12	
Kapitza's Pendulum	13	
Relativistic Particle in a Box	10	
Ion Trap Chips	15	
Hyperfine Qubits in Trapped Neutral Atoms	10	
Schrödingers Cat	10	
Boltzmann Machine	8	
The Precession of Mercury	14	
Pulsar Electrodynamics	10	
Total	**102**	

Dear contestants, Welcome to PLANCKS Austria 2025,

- The language used in this competition is **English**, but you may also provide your answers in **German**.
- Write **each problem on a separate sheet**, **number the pages**, and include your (abbreviated) **team name on each sheet**. You may of course use more than one sheet per example.
- A transparent and well-organized approach is recommended. Additionally, some sub-points can be solved independently of previous tasks — so don't give up too early, and give it a try!
- When a problem is unclear, a participant can ask, via the crew, for a clarification. If the response is relevant to all teams, the jury will provide this information to the other teams.
- You are allowed to use a **non-programmable, not-graph calculator** (But scientific is okay).
- **No books or other sources**, except for this exercise booklet and a dictionary, are to be consulted during the competition.
- The organization has the right to disqualify teams for misbehavior or breaking the rules.
- The use of hardware (including phones, tablets, etc.) is not permitted, except for (non-smart) watches and medical equipment. Phones must be stored away and should not be kept in pockets.
- In situations to which no rule applies, the organization decides. We wish you all the very best.

May the best physics team win!

1 Problem Quartet

12 points *Matthias Diez*, TU Graz & KFU

1.1 Bouncing Balls

Consider the setup as illustrated in Figure 1.1.

(a) **(1.5 points)** Start with two balls on top of each other(see figure), with $m_1 > m_2$. They will fall down from a height h with velocity v(0) = 0. Determine the velocity v_2 of ball 2 after bouncing off the floor in dependence of the mass $a = \frac{m_2}{m_1}$.

(b) **(0.5)** What is the maximum velocity ball 2 can reach in dependence of this mass ratio.

(c) **(0.5)** At what mass ratio ball 1 comes to rest after bouncing off the floor and determine the maximum height ball 2 can reach in this case.

(d) **(0.5)** Now add a third ball with mass m_3, and consider $m_1 \gg m_2 \gg m_3$. What is the maximum height ball m_3 can reach, after bouncing off the floor. The radius of ball 2 is r_2.

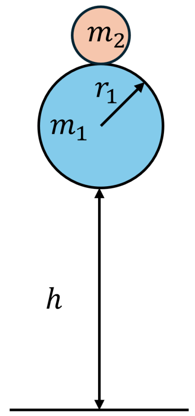


Figure 1.1: Bouncing Balls

1.2 Falling Conductor Loop

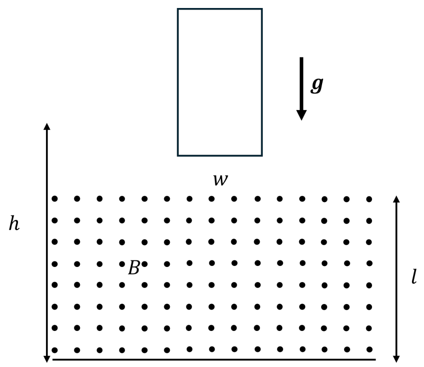


Figure 1.2: A conductor loop falling into a homogeneous magnetic field

(3 points) A rectangle conductor loop with length l, width w and resistance R is falling in earth's gravitational field, from an initial height h with an inital velocity $v(0) = 0$. Right above earth's surface is a homogeneous magnetic field with strength B, pointing in a direction perpendicular to the conductor loop. The magnetic field extends to a height l. Calculate the time dependence of the velocity of the conductor loop inside the magnetic field.

1.3 Fata Morgana

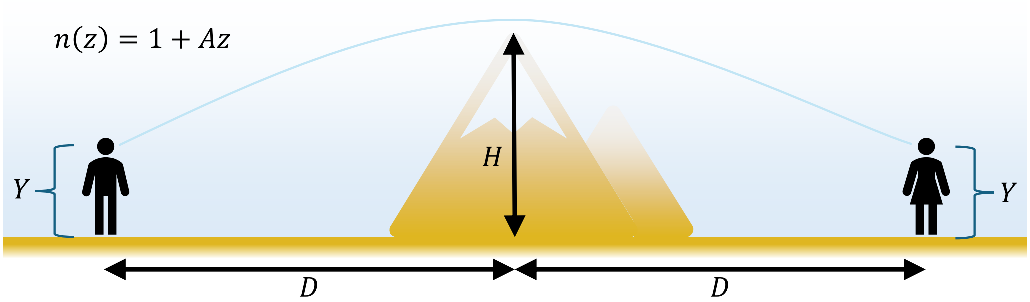


Figure 1.3: Fata Morgana path of light.

(3 points) A man with eye level Y stands in the dessert at a distance D from an object of height H. On the opposite sight stands his girlfriend, whose eyes are on the same height above the floor. Due to the heat the refraction index of air changes approximately as $n(z) = n_0(1 + Az)$. Determine conditions on A (You do not need to solve for A), such that they can look each other in the eye when both look up in an angle θ_0. θ_0 is the angle between the vertical axis and the light ray. Furthermore determine the trajectory of light $z(x)$ where z is the height of the light above the floor, for given A.

1.4 Water reservoir

(3 points) At the bottom of a water tower, with a conic water reservoir, is a small hole with radius $r \ll R$. The radius of the cone depends on the height $R(h) = kh$. At $t = 0$, the water is at height h. Determine the time evolution of the water beam radius $b(t)$ at a distance d under the reservoir, long before the reservoir is empty.

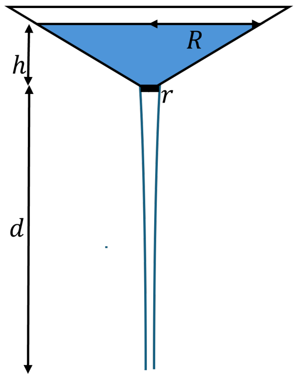
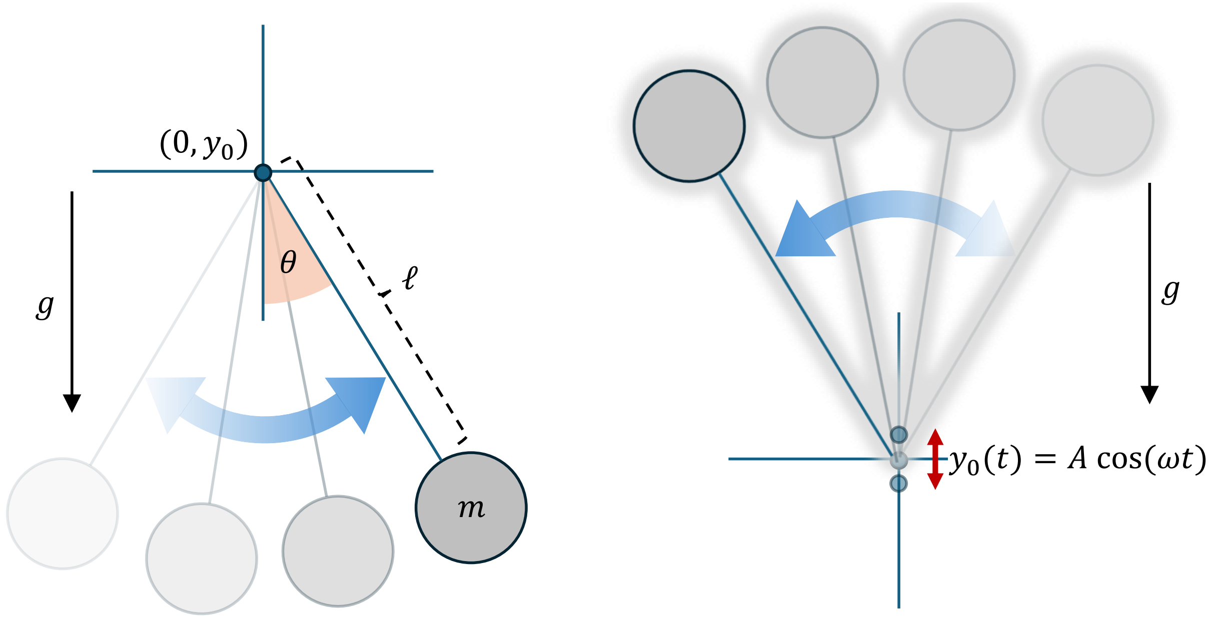


Figure 1.4: Water tower

2 Kapitza's Pendulum

13 points *Johannes Krondorfer*, TU Graz



(a) Coordinates (b) Dynamic Stabilization

Figure 2.1: Definition of the coordinates for Kapitza's pendulum (a) and illustration of dynamical stabilization for a fast oscillating pivot point (b).

Time-dependent problems pose a significant challenge in both classical and quantum mechanics. Unlike time-independent systems, where well-established techniques allow us to determine solutions analytically, time-dependent systems often require intricate methods or purely numerical approaches. However, in some cases, the specific structure of time dependence allows for systematic analytical treatments.

In this example, we explore **Kapitza's pendulum** – a simple pendulum of mass m, with a massless rod of length ℓ in a homogeneous gravitation field $\boldsymbol{g} = g\,\hat{e}_y$, with a periodically oscillating pivot point $(0, y_0(t)) = (0, A\cos(\omega t))$, as illustrated in Figure 2.1a. General goal of this example is to analyze the stability of the pendulum for a fast oscillating pivot point with small amplitude, where we can observe the phenomenon of dynamical stabilization, as illustrated in Figure 2.1b. To this end, we will derive and analyze the equations of motion and employ **Floquet theory** and the **Magnus expansion**, two essential tools in studying time-periodic ordinary differential equations (ODEs). Although we investigate a classical system, the investigated methods are broadly applicable to periodically driven quantum systems, where they unveil insights into phenomena such as dynamical stabilization, topological phases, and coherent control.

2.1 Equations of Motion

Consider a pendulum of length ℓ and mass m suspended from a pivot at $(0, y_0)$ as depicted in Figure 2.1.

(a) **(0.5 points)** For a fixed pivot point $y_0 = \text{const.}$ derive the equation of motion for the angle θ.

(b) **(0.5 points)** Find the stationary points of the equation of motion. Linearize the equation around these points and analyze their stability. If the solution is stable, calculate the oscillating frequency ω_0.

(c) **(1 point)** Now, consider a time-dependent pivot $y_0(t)$. Derive the modified equation of motion for θ. Compare this to part (a) and provide a physical interpretation of additional terms.

(d) **(0.5 points)** Linearize the equation around the stationary points and express it as a first-order vector valued ODE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ \alpha_\pm(t) & 0 \end{bmatrix} \boldsymbol{x}\,. \tag{2.1}$$

2.2 Floquet-Lyapunov Theorem

For the remaining problem we will consider the specific case of $y_0(t) = A\cos(\omega t)$, and derive stability porperties of this system. Generally, linear time-dependent ODEs, such as (2.1), are not analytically solvable. However, for linear ODEs, with time-periodic matrix, a specialized treatment is possible, as shown by the following theorem.

Theorem (Floquet-Lyapunov Theorem)**.** *Consider the system of linear differential equations*

$$\frac{d}{dt}\boldsymbol{x}(t) = H(t)\boldsymbol{x}(t), \tag{2.2}$$

where $H(t)$ is a time-periodic continuous matrix function with period T, i.e.,

$$H(t+T) = H(t), \quad \textit{for all } t\,. \tag{2.3}$$

Then, the fundamental solution matrix (the propagator) $U(t)$[1] *, i.e. the solution to the matrix differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H(t)U(t) \quad \textit{with} \quad U(0) = I\,, \tag{2.4}$$

can be expressed as

$$U(t) = P(t)e^{\widetilde{H}t}\,, \tag{2.5}$$

where $P(t)$ is a T-periodic matrix, i.e. $P(t+T) = P(t)$ and $\widetilde{H} = \frac{1}{T}\log U(T)$ is a constant matrix.

[1]For a linear first order ODE the propagator $U(t)$ can be used to determine the solution for given initial conditions $x(0) = x_0$, by $x(t) = U(t)\,x_0$. Thus the propagator completely determines the time evolution of the system. While in classical mechanics this concept is not necessarily introduced, the propagator, or time-evolution operator has an essential role in quantum mechanics and the mathematical treatment of ODEs.

(a) **(2 point)** Prove the Floquet-Lyapunov theorem, i.e. show (2.5).[2]

(b) **(2 point)** Perform the variable transformation $W(t) = P(t)^{-1}U(t)$ to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \widetilde{H}W(t) \tag{2.6}$$

and conclude that the long-term evolution and stability is determined by $\widetilde{H}$ instead of P. Under what conditions is the solution stable?[3]

2.3 Dyson Series and Magnus Expansion

Now we know which time-independent quantity is of interest to us for determining the stability of the Kapitza pendulum. However, we still cannot compute $U(T)$ analytically, but need to employ approximations. The standard method for perturbative expansion of differential equations of the form (2.4) is the so called Picard iteration or Dyson series, where we write

$$U(t) = \sum_{k=0}^{\infty} U^{(k)}(t)\,, \tag{2.7}$$

with $U^{(k)}(t) = \mathcal{O}(||H||^k)$ and $U^{(0)} = I$.

(a) **(1 point)** Show that this ansatz (2.7) yields an order by order representation with

$$U^{(k)}(t) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \;\cdots\; \int_0^{t_{k-1}} \mathrm{d}t_k\; H(t_1)H(t_2)\ldots H(t_k)\,. \tag{2.8}$$

(b) **(1 point)** Show that if H commutes for different times, i.e. $[H(t), H(t')] = 0$ for all t and t', then $U(t)$ can be written via the exponential

$$U(t) = \exp\left(\int_0^t H(t')\,\mathrm{d}t'\right), \tag{2.9}$$

and argue why this is not the case for (2.1).

In general, these perturbative approach is not optimal, since important properties of the system might be not conserved, such as symplecticity in classical hamiltonian systems or unitarity in quantum mechanical systems. Therefore a different approach is more promising to obtain better and more physical approximation of the system. This leads to the concept of Magnus expansion.

In the Magnus Expansion the evolution operator $U(t)$ of the system is expressed as a proper matrix exponential by defining $\Omega(t) := \log U(t)$ and thus $U(t) = \exp(\Omega(t))$. Assuming that Ω can be written as an infinite sum in orders of H, we write $\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$ with $\Omega_k = \mathcal{O}(||H||^k)$.

(c) **(1 point)** Show that Ω_1, Ω_2 and Ω_3 can be expressed as[4]

$$\begin{aligned}
\Omega^{(1)}(t) &= U^{(1)}(t)\\
\Omega^{(2)}(t) &= U^{(2)}(t) - \frac{1}{2}(\Omega^{(1)}(t))^2\\
\Omega^{(3)}(t) &= U^{(3)}(t) - \frac{1}{2}\left(\Omega^{(1)}(t)\Omega^{(2)}(t) + \Omega^{(2)}(t)\Omega^{(1)}(t)\right) - \frac{1}{6}(\Omega^{(1)}(t))^3\,.
\end{aligned} \tag{2.10}$$

[2] *Hint:* The differential equation (2.4) with initial conditions uniquely defines the propergator U. It might be helpful to prove $U(t+T) = U(t)U(T)$ first.

[3] *Hint*: Think about the boundedness of P and the eigenvalues of $\widetilde{H}$.

[4] *Hint*: Use Taylor expansion and gather the terms of the same order.

2.4 Stability Analysis of Kapitza's Pendulum

With these theoretical results we can now analyze the stability of Kapitza's pendulum in linearized form (see (2.1)) with a time-dependent pivot point $y_0(t) = A\cos(\omega t)$ with fast oscillation frequency ω of the pivot point, and small amplitude A. With the previous sections we know that the long term evolution is governed by $\widetilde{H}$ by Floquet analysis and with the Magnus expansion we have that $\widetilde{H} = \frac{1}{T}\Omega(T)$. So let's determine the effective (averaged) evolution of Kapitza's pendulum and determine its stability.

(a) **(2 points)** Show that the effective time-independent evolution matrix for and $A, \frac{1}{\omega} \ll 1$ up to third order is given by

$$\widetilde{H}_\pm = \frac{\Omega_\pm(T)}{T} \approx \frac{\Omega_\pm^{(1)}(T) + \Omega_\pm^{(2)}(T) + \Omega_\pm^{(3)}(T)}{T} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}\left(\frac{A\omega}{\ell}\right)^2 \mp \omega_0^2 & 0 \end{bmatrix}, \tag{2.11}$$

where $T = \frac{2\pi}{\omega}$ is the period of the oscillating pivot point.[5] Note that $\Omega_\pm^{(2)}(T) = 0$ and you do not need to calculate this term.

(b) **(1.5 point)** Convert the effective first-order ODE obtained above back into a second-order equation to obtain the effective second order differential equation. Analyze the stability of both stationary points and interpret the results. Derive conditions on the stability and compute the frequency of the pendulum.

[5] *Hint*: You need to neglect small terms of $A, \frac{1}{\omega} \ll 1$ to get the result. You can neglect terms of $A, A\omega_0, A\omega_0^2$ for small amplitudes A and fast frequencies ω. Some terms cancel each other anyway and some have to be neglected. Note that you cannot neglect $A\omega$.

3 Relativistic Particle in a Box

10 points *Martin Napetschnig*, TU Munich

[1, 2, 3, 4] The particle in a box is a well-known problem of non-relativistic quantum mechanics that you are probably familiar with from your courses. In this exercise, you will work out the treatment of a relativistic particle in a box. You will first derive a relativistically covariant evolution equation, the *Klein-Gordon equation*. Then, you have to settle whether a 'box', i.e. an infinite square well, can actually contain one and only one particle forever. This seemingly stupid question caused intense debates among the most brilliant minds back in the 1930s, including Niels Bohr, Arnold Sommerfeld and Fritz Sauter, after Oscar Klein brought up a famous paradox - the *Klein paradox* - according to which high potential barriers for relativistic particles **seem to reflect more particles than are incoming!**

You can keep or skip all factors of the speed of light c in your calculation, but do it consistently! The Lagrange function of a relativistic point particle in one dimension for a spinless particle with charge q and mass m is given by

$$L(x,\dot{x}) = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - V(x)\,. \tag{3.1}$$

(a) **(1 point)** To verify the non-relativistic limit, derive the equations of motion to first order in $\left(\frac{\dot{x}}{c}\right)^2$. You should recover a familiar result.

(b) **(1.5 points)** Find the Hamilton function $H(p,x)$. For $p = 0 = V$ you should recover another familiar result.

(c) **(0.5 points)** With the Hamiltonian found we now consider operators $H \to \hat{H} = i\hbar\frac{\partial}{\partial t}$, $x \to \hat{x}$, $p \to \hat{p} = -i\hbar\frac{\partial}{\partial x}$. It is more convenient to work with the squared version of the Hamiltonian for the quantum mechanical evolution equation. Show that the *squared Schrödinger equation* is given by

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{V}(\hat{x})\right)^2 \Psi(t,x) = \left(-\hbar^2c^2\frac{\partial^2}{\partial x^2} + m^2c^4\right)\Psi(t,x) \tag{3.2}$$

(d) **(0.5 points)** From now on consider the case of an electrostatic potential $V(x) = qV_0 = const$. Show that

$$\Psi(t,x) = e^{-\frac{i}{\hbar}Et}\left(Ae^{\frac{i}{\hbar}kx} + Be^{-\frac{i}{\hbar}kx}\right) \tag{3.3}$$

is a solution to (3.2), where A, B, k do not depend on x or t. Find the expression for k.

Now consider the situation of a particle moving from left to right, scattering off a potential well of height qV_0, as sketched in Figure 3.1. Depending on the hierarchy between E & V_0, in region II there are 3 different regimes

- Weak potential: $E > qV_0 + mc^2$
- Intermediate potential: $qV_0 - mc^2 < E < qV_0 + mc^2$
- Strong potential: $E < qV_0 - mc^2$

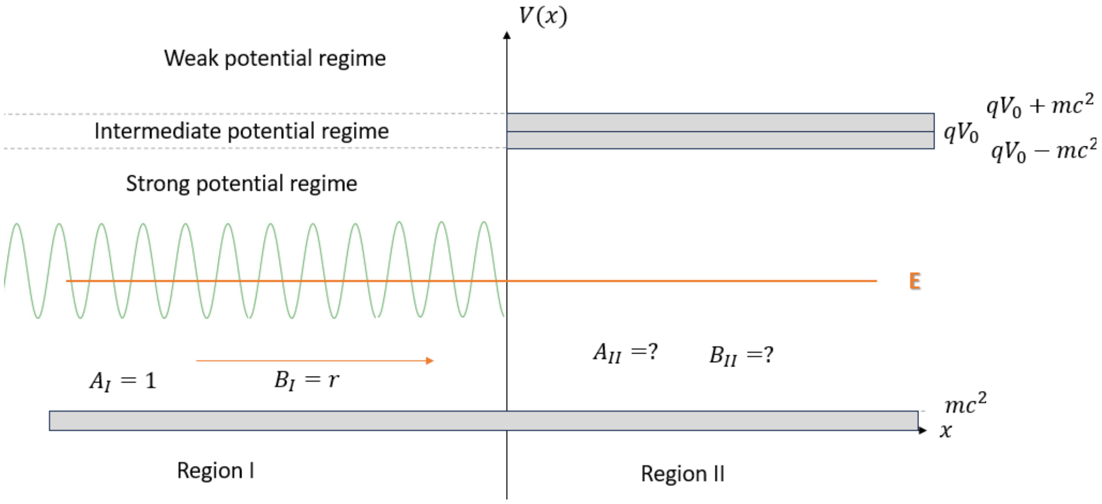


Figure 3.1: Particle scattering off the potential well of height $qV_0, V_0 > 0$. The energy of the particle is positive and the particle is initially moving from left to right.

(e) **(2 points)** The terms containing the coefficients A and B in (3.3) describe left- and right-moving states. For each of the three listed regimes, write down whether A_{II} and B_{II} multiply with left-moving states, right moving states or exponentially decaying states[6].[7]

(f) **(2 points)** Given that $\Psi(t,x)$ and $\frac{d}{dx}\Psi(t,x)$ must be continuous at the well, find the matching relations between $A_{\text{I}}, B_{\text{I}}, A_{\text{II}}$ and B_{II} in the **strong potential** regime and derive the reflectivity and transmissivity r and t. You should find 2 equations for 2 unknowns.[8]

(g) **(1.5 points)** Calculate the reflection coefficient $R = |r|^2$ and the transmission coefficient $T = 1 - R$. Do your results make sense? Do you have an idea how to resolve the paradox? You may notice that $T \neq |t|^2$. Does this surprise you? Give expressions for R and T in the limit $V_0 \to \infty$.

(h) **(1 point)** As you should have found in point 7, an infinitely high barrier is fully reflective and conserves particle number, thus making our box 'save'. Consider now a particle trapped in such a box, as sketched in fig. 3.2. Find the eigenfunctions and eigenenergies of the system. Make a suitable Taylor expansion of the eigenenergies to recover the non-relativistic energy levels for a particle in a box $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

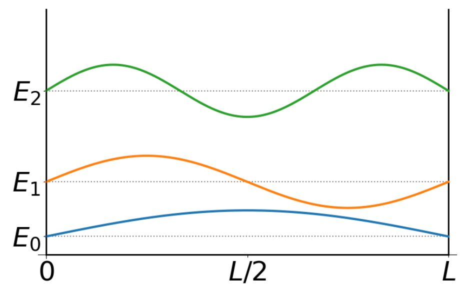


Figure 3.2: In the limit $V_0 \to \infty$, a particle inside the potential well is reflected from the walls and thereby stays in the box, fixing the boundary conditions.

[6]Exponentially growing states are unphysical because $\Psi(t,x)$ would not be normalizable.

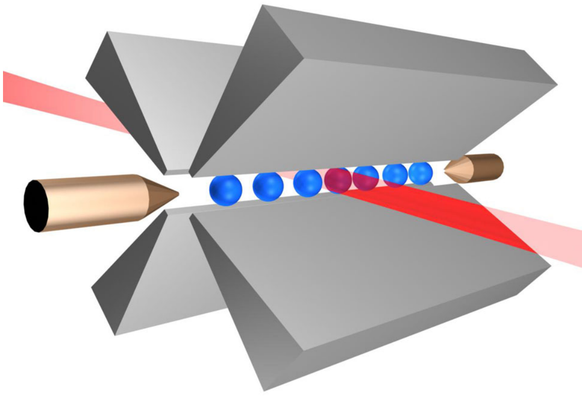
[7]*Hint*: A left-moving state is a state for which the group velocity $v_G := \frac{\partial E}{\partial p}$ is negative, while for right-moving states it is positive.

[8]*Hint*: In region I, you can normalize $A_{\text{I}} = 1$ and interpret $B_{\text{I}} \equiv r$ as the amplitude of the wave reflected from the well. In region II, you can set the coefficient for the left-moving wave found in point (e) to zero (because there should be no particle flux from right to left in region II), while the other coefficient can be interpreted as the transmittivity t.

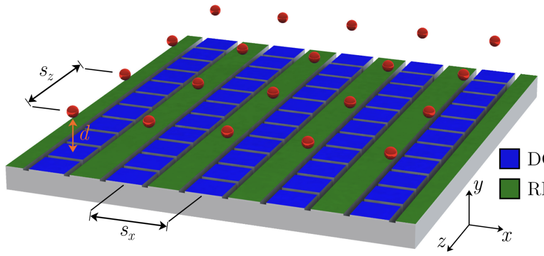
4 Ion Trap Chips

15 points *Michael Pfeifer*, Universität Innsbruck

Ion traps are devices used to confine electrically charged particles. In this exercise, we will look at Paul traps that use (partially time-dependent) electrical fields for ion confinement. Micro-fabricated ion traps (ion trap chips) lie at the heart of some of the world's most advanced quantum computers. In this exercise, we will explore some of the fundamentals of macroscopic Paul traps and ion trap chips.



(a) Drawing of a macroscopic Paul trap with ions (blue) and a laser beam (red). From: https://www.uibk.ac.at/exphys/qo/research/trappedions



(b) Drawing of an ion trap chip from: P. Holz *et al.*, Adv. Quantum Technol. **3** (2020).

Figure 4.1: Macroscopic Paul trap and ion trap chip.

4.1 Macroscopic Paul Traps

(a) **(1 point)** Show that it is not possible to stably confine an electrically charged particle using only electrostatic fields, i.e. it cannot be maintained in a stable stationary equilibrium using only electrostatic fields. This result is called Earnshaw's theorem.

We have shown in the previous exercise, that it is impossible to confine ions using only electrostatic fields. But we can do so using alternating electric fields, possibly in combination with static electric fields.

(b) **(2 points)** Consider the four infinitely long rods in figure 4.2 with linear charge densities $\pm\lambda(t) = \pm\lambda_0 \cos \Omega t$. Calculate the electrical potential $\varphi(t, x, y)$ close to the center up to second order in x, y. Assume $R \ll d$.

(c) **(0.5 points)** Derive the classical equations of motion of an ion of charge e and mass m in the electric potential $\varphi(t, x, y)$ from above, neglecting any motion in the z-direction. For the setting in figure 4.2, analyze the equations of motion for $\Omega = 0$. Are charged particles in the center confined in the x and y directions in this case?

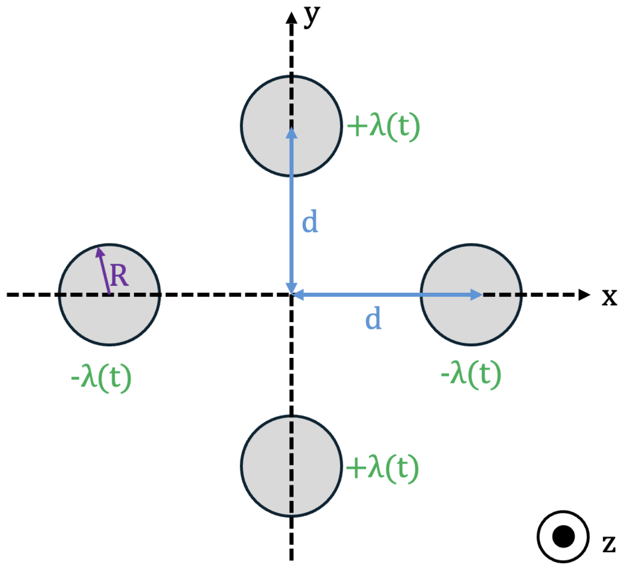


Figure 4.2: Four infinitely long electrically charged rods extending in z-direction with line charges $\pm\lambda(t)$.

The alternating electric field $\varphi(t,x,y) = \varphi(x,y)\cos\Omega t$ leads to a confining force on a charged particle of charge e and mass m in the x,y-plane. This confining force can be described by an effective potential

$$\psi(x,y) = \frac{e}{4m\Omega^2}|\vec{\nabla}\varphi(x,y)|^2. \tag{4.1}$$

(d) **(0.5 points)** Write down the quantum mechanical Hamiltonian of an ion of charge e and mass m in the potential $\phi_{\text{tot}}(x,y) = \psi(x,y)$, using the $\varphi(x,y)$ found in sub-problem (b), and using ladder operators $a_i, a_i^\dagger$ for the vibrational degrees of freedom. Neglect again any motion in the z-direction. Assume the internal degrees of freedom of the ion to be that of a two-level system. What are the oscillation frequencies?

4.2 Micro-fabricated Ion Traps

It turns out that ions cannot only be trapped using macroscopic Paul traps with three-dimensional electrode arrangements, but also using electrodes arranged in a plane. This opens up the exciting possibility of micro-fabricating ion traps using standard CMOS fabrication techniques.

(a) **(4 points)** Consider an infinitely long electrode extending between $x_1 \leqslant x \leqslant x_2$ in the x,z-plane, shown in figure 4.3. The electrode extends between x_1 and x_2 in x-direction and from $-\infty$ to $+\infty$ in the z-direction. Assume that the electrode is on the potential V and the rest of the x,z-plane is grounded. Calculate the electrostatic potential $\varphi(x,z,y)$ for $y > 0$.[9] The following integrals might be useful:

$$\int \frac{1}{(x^2+y^2+(a-z)^2)^{3/2}}\mathrm{d}z = \frac{z-a}{(x^2+y^2)\sqrt{(a-z)^2+x^2+y^2}} + const. \tag{4.2}$$

[9] *Hint*: You may for example use Green's functions or a Fourier transform in x, to solve for φ. Also think about the homogeneity in z.

$$\int \frac{y}{(a-x)^2+y^2}\mathrm{d}x = \arctan\left(\frac{x-a}{y}\right) + const. \tag{4.3}$$

$$\int_{-\infty}^{\infty} \frac{e^{ika}}{ik} e^{-|k|y} dk = -2\arctan\left(\frac{a}{y}\right) + \pi \tag{4.4}$$

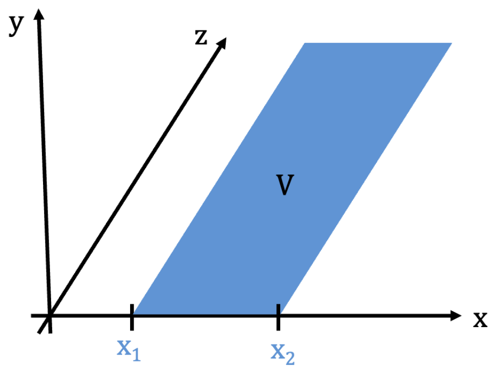


Figure 4.3: Electrode on voltage V extending in the x, z-plane, with the rest of the plane grounded.

(a') ***(1 point)*** *If you cannot derive the potential $\varphi(x, y)$ in part (a), you can request the solution. To achieve the point you have to show that it is indeed the correct potential satisfying the boundary condition.*

(b) **(0.5 points)** Assume that the two infinitely long electrodes in the configuration in figure 4.4 are on the voltage $V(t) = V_0 \cos \Omega t$ and the rest of the x, z-plane is grounded. The electrodes extend between $-\infty$ and $+\infty$ in the z-direction and between $-c \leqslant x \leqslant 0$ and $a \leqslant x \leqslant a + b$ in the x-direction. This configuration is an example for the rf electrodes of an ion trap chip. Calculate the electric potential $\varphi(x, y, t)$ of this configuration for $y > 0$. (Neglect for all calculations the z-direction.)

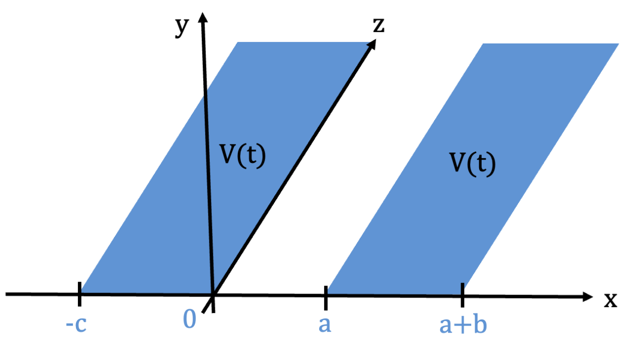


Figure 4.4: Two rf electrodes in the x, z-plane, with the rest of the plane grounded.

(c) **(1 point)** The effective potential $\psi(x, y)$ in configuration of the previous sub-exercise has a minimum at (x_0, y_0), with $x_0 = ac/(b + c)$ and $\psi(x_0, y_0) = 0$. For the case $b = c$, calculate the ion-surface distance y_0. How can you change the ion-surface distance? (Neglect for all calculations the z-direction.)

One can write down the formula for the electric potential on such an ion trap chip around the point (x_0, y_0) in the form:

$$\varphi(x,y) \approx \varphi_0 + \frac{\kappa V_0}{2y_0^2}\left(x^2 - y^2\right)\cos\Omega t, \tag{4.5}$$

where y_0 is the distance of the minimum from the chip surface. The parameter κ is called the trap efficiency.

(d) **(1 point)** Determine κ for the given configuration for $b = c$.

4.3 Double-wells

By using 3 rf electrodes, one can create a (radial rf-) double-well potential in the x, y-plane above an ion trap chip.
Consider in one dimension (x), in the two separate wells of a double-well potential, two quantum mechanical, charged particles A and B. They have charge Q_A and Q_B, respectively, and mass m. In equilibrium, they have a distance s_0 from each other and oscillation frequencies ω_A and ω_B at their respective potential minima.

(a) **(2.5 points)** Determine the coupling strength $\hbar\Omega_\mathrm{ex}$ between the two particles A and B to lowest order.
Note: Use the "rotating wave approximation", i.e. $a_i^\dagger a_i^\dagger = 0$, $a_i a_i = 0$, where the $a_i^\dagger, a_i$ $(i = A, B)$ are the creation and annihilation operators of phonons in the potential minima.

(b) **(0.5 points)** Write down the quantum mechanical Hamiltonian of this double-well system using ladder operators $a_i, a_i^\dagger$, considering only the phononic modes of the double wells and ignoring the internal degrees of freedom of the trapped charged particles.

As a model of the interaction of two trapped particles in a double-well potential, consider now two classical particles/blocks A and B, both of mass m, as shown in figure 4.5. They are coupled to walls with springs of coupling constants k_A and k_B and to each other with a spring of coupling constant c.

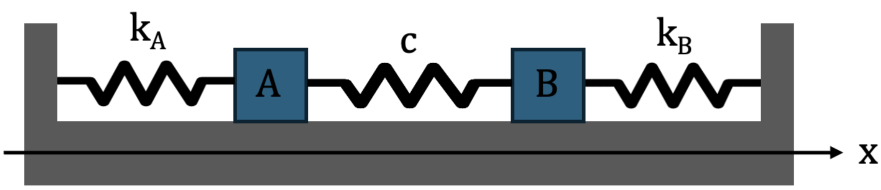


Figure 4.5: Two blocks A, B with springs.

(c) **(0.5 points)** Calculate the eigenfrequencies.

(d) **(1 point)** Consider the case $k_A = k_B = k$. Assume that the particle A is displaced by $x_\mathrm{A}(0) = A_0$ at $t = 0$, while the other particle is initially at rest, and then let to oscillate. After which time t_ex is the kinetic energy fully transferred from particle A to particle B for the first time?

5 Hyperfine Qubits in Trapped Neutral Atoms

10 points

Johannes Krondorfer, TU Graz

Neutral atom traps are among the most promising platforms for quantum computing. In these systems, qubits are typically encoded in the fine or hyperfine structure of the atom, taking advantage of their long-lived states and well-characterized interactions with external fields. Their precise level structure and weak environmental coupling make them ideal candidates for applications ranging from quantum computing to atomic clocks and quantum simulations. Here, we will consider simple two and three level models to describe the basic principles of optical dipole traps and single qubit operations for hyperfine (i.e. nuclear spin) qubit encoding in alkaline earth or alkaline earth-like atoms, such as ^{87}Sr or ^{171}Yb.

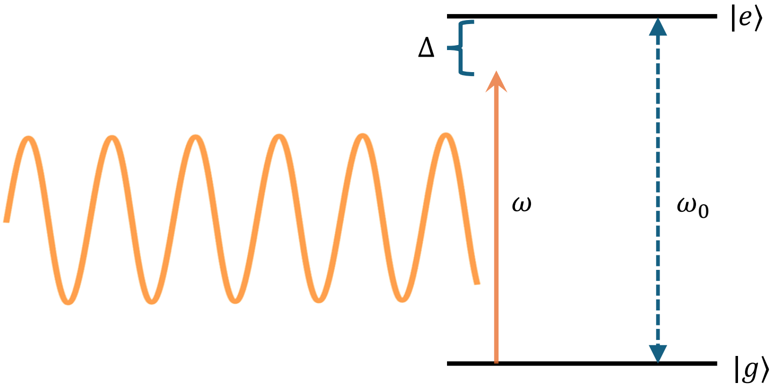


Figure 5.1: Two-Level system in an external field. Definition of variables and states.

5.1 A Two Level Atom in a Laser Field

To warm up, we consider a simple two-level atom in a laser field and study the dynamics for different laser parameters. The general internal Hamiltonian of the two-level system $\{|g\rangle, |e\rangle\} \hat{=} \{[0 \quad 1]^T, [1 \quad 0]^T\}$ in dipole approximation is given by

$$H = \hbar\omega_0\sigma^\dagger\sigma - \left(\boldsymbol{\mathcal{D}}\sigma^\dagger + \boldsymbol{\mathcal{D}}^*\sigma\right)\cdot \boldsymbol{E}(x,t)\,, \tag{5.1}$$

with the atomic lowering operator $\sigma = |g\rangle\langle e| \hat{=} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, the dipole element $\boldsymbol{\mathcal{D}} = \langle e|\,\hat{\boldsymbol{d}}\,|g\rangle$ and the electric field amplitude $\boldsymbol{E}(x,t)$. An illustration is provided in Figure 5.1. For now, we assume that the atomic motion can be neglected and approximate the electric field as space independent, i.e. $\boldsymbol{E}(x,t) = \boldsymbol{E}(t) = \boldsymbol{E}_0\cos(\omega t)$.

(a) **(1.5 points)** Transform the Hamiltonian (5.1) into the rotating frame by applying the unitary transformation $V(t) = \mathrm{e}^{\mathrm{i}\omega t\sigma^\dagger\sigma}$. Apply the rotating wave approximation (RWA), i.e. $1 + \mathrm{e}^{\pm 2\mathrm{i}\omega t} \approx 1$, to obtain a time independent Hamiltonian of the form

$$H' = -\hbar\Delta\sigma^\dagger\sigma + \frac{\hbar}{2}\left(\Omega\sigma^\dagger + \Omega^*\sigma\right)\,, \tag{5.2}$$

and determine the parameters Δ and Ω.[10]

[10] *Hint:* Think about how to correctly apply a time dependent unitary transformation to the Hamiltonian. Note that the time dependent schrödinger equation has to be satisfied for the transformed state $|\psi'\rangle = V(t)\,|\psi\rangle$.

(b) **(0.5 points)** Rewrite the Hamiltonian as a linear combination of Pauli matrices[11] and the unit matrix, i.e.

$$H' = \alpha \mathbb{1} + \frac{\hbar\tilde{\Omega}}{2}\frac{\tilde{\boldsymbol{\Omega}}}{\tilde{\Omega}} \cdot \boldsymbol{\sigma}\,, \tag{5.3}$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

(c) **(1 point)** Show that for a unit vector $\boldsymbol{n}$ the following relation holds for the Pauli matrices[12]

$$\exp(\mathrm{i}\theta\, \boldsymbol{n}\cdot\boldsymbol{\sigma}) = \cos(\theta)\mathbb{1} + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\theta)\,. \tag{5.4}$$

(d) **(1 point)** Compute the time evolution operator $U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H't}$ and the excited state population $\rho_{ee}(t) = |\langle e|\, U(t)\, |\psi_0\rangle|^2$ for the initial state $|\psi_0\rangle = |g\rangle$. Sketch it for $\Delta = 0, \Omega/2, \Omega, 2\Omega, 5\Omega$.

5.2 Optical Lattices

After the atoms have been pre cooled using techniques like Doppler cooling and magneto optical traps, they can be loaded into optical lattices, where the atoms see a confining potential simply by interacting with the laser field. To describe this behavior we have to introduce the spatial degrees of freedom and write the Hamiltonian as

$$H = \frac{p^2}{2m} - \hbar\Delta\sigma^\dagger\sigma + \frac{\hbar}{2}\left(\Omega(x)\sigma^\dagger + \Omega(x)^*\sigma\right)\,, \tag{5.5}$$

where the state vector is now written as $|\psi\rangle = \psi_e(x,t)\,|e\rangle + \psi_g(x,t)\,|g\rangle$. For simplicity, we consider only one spatial dimension.

(a) **(0.5 points)** Derive the Schrödinger equation for the coefficients $\psi_e(x,t)$ and $\psi_g(x,t)$.

(b) **(1 point)** Apply an adiabatic approximation ($\partial_t\psi_e(x,t) \approx 0$, $\Delta \pm \frac{p^2}{2m} \approx \Delta$) for the excited state population to obtain an effective Hamiltonian for the ground state evolution

$$H_{\text{eff}} = \frac{p^2}{2m} + V_{\text{eff}}(x)\,. \tag{5.6}$$

(c) **(2 points)** For two red detuned ($\Delta < 0$) counter propagating laser beams

$$E(x,t) = E_+(x,t) + E_-(x,t) = \frac{E_0}{2}\left(\cos(kx - \omega t) + \cos(kx + \omega t)\right) = E_0\cos(kx)\cos(\omega t)\,,$$

we obtain a standing wave as trapping potential, i.e. $\Omega(x) = \Omega_0\cos(kx)$ as illustrated in the left part of Figure 5.2.[13]

(i) Estimate the number of vibrational levels of a trapped atom.

(ii) How must the laser beams be selected to obtain an effective potential moving with constant velocity v.

(iii) How must the laser beams be selected to obtain an effective potential that is accelerating.

(iv) Estimate the maximal acceleration such that the atoms remain trapped.

---

[11] The Pauli matrices are given by $\sigma_x = \sigma + \sigma^\dagger \mathrel{\hat{=}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \mathrm{i}\sigma - \mathrm{i}\sigma^\dagger \mathrel{\hat{=}} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$, $\sigma_z = \sigma^\dagger\sigma - \sigma\sigma^\dagger \mathrel{\hat{=}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

[12] The (anti-)commutation relations for the Pauli matrices are $[\sigma_n, \sigma_m] = 2\mathrm{i}\epsilon_{nmk}\sigma_k$ and $\{\sigma_n, \sigma_m\} = 2\delta_{nm}$.

[13] *Hint*: You may use the adiabatic approximation $E(x,t) \approx E_0\cos(k(t)x - \int_0^t \omega(t')\,\mathrm{d}t') \approx \cos(kx - \int_0^t \omega(t')\,\mathrm{d}t')$ which is valid if $\dot{\omega} \ll \omega^2$ and $\delta\omega/c \ll \omega/c = k$.

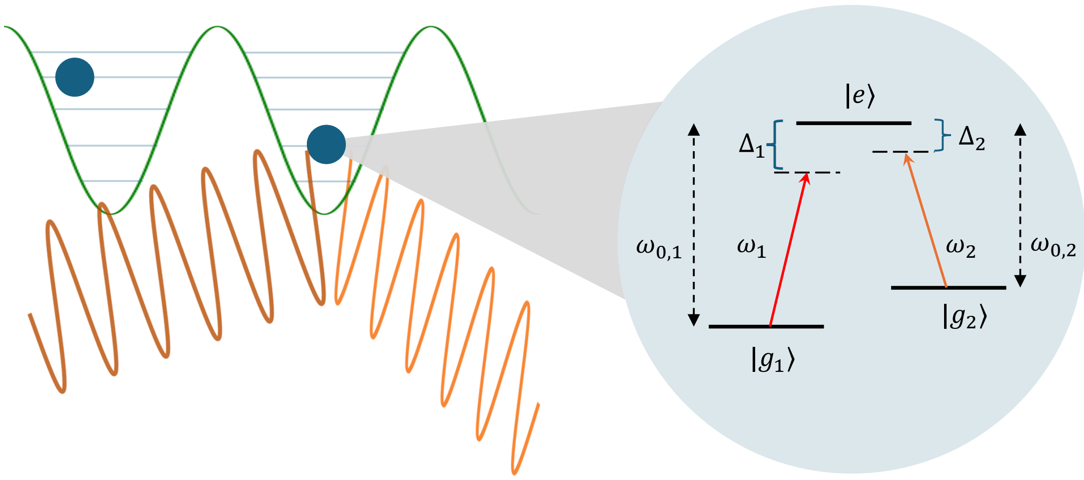


Figure 5.2: Illustration of standing wave potential for trapped atoms (left) and internal three level structure (right), with hyperfine ground states $|g_1\rangle$, $|g_2\rangle$ and excited state $|e\rangle$.

5.3 Hyperfine Transitions for Trapped Atoms

Now that we have trapped the atoms in an optical lattice (see Figure 5.2, we need schemes for how to manipulate the hyperfine states, in which we want to encode a qubit. We only discuss the realization of single qubit operations, as two qubit gates are theoretically more challenging. The problem for hyperfine states is that transitions cannot be driven directly by optical laser fields due to the small energy splitting in the electronic ground state and the violation of optical selection rules. However, transitions in the hyperfine levels (nulcear spin states) of the electronic ground state can be driven by two-photon Raman processes, whose principle we will derive below.

We consider the three level system $\{|g_1\rangle, |g_2\rangle, |e\rangle\}$ as depicted on the right side of Figure 5.2. Assume that two laser fields are applied, one coupling $|g_1\rangle$ to $|e\rangle$, the other coupling $|g_2\rangle$ to $|e\rangle$. Setting the energy of the excited state $|e\rangle$ to zero, the Hamiltonian is then given by

$$H = -\hbar\omega_{0,1}|g_1\rangle\langle g_1| - \hbar\omega_{0,2}|g_2\rangle\langle g_2| - \left(\boldsymbol{\mathcal{D}}_1\sigma_1^\dagger + \boldsymbol{\mathcal{D}}_1^*\sigma_1\right)\cdot\boldsymbol{E}_1(x,t) - \left(\boldsymbol{\mathcal{D}}_2\sigma_2^\dagger + \boldsymbol{\mathcal{D}}_2^*\sigma_1\right)\cdot\boldsymbol{E}_2(x,t)\,, \tag{5.7}$$

with $\sigma_\alpha = |g_\alpha\rangle\langle e|$. Let's assume $\boldsymbol{E}_1(x,t) = \boldsymbol{E}_{0,1}\cos(\omega_1 t)$ and $\boldsymbol{E}_2(x,t) = \boldsymbol{E}_{0,2}\cos(\omega_2 t)$. Via unitary transformations $V_1(t) = \mathrm{e}^{-\mathrm{i}\omega_1 t|g_1\rangle\langle g_1|}$ and $V_2(t) = \mathrm{e}^{-\mathrm{i}\omega_2 t|g_2\rangle\langle g_2|}$ we transform into the respective rotating frame and by applying the rotating wave approximation we obtain the Hamiltonian

$$\begin{aligned} H' = &-\hbar\Delta|e\rangle\langle e| + \hbar(\Delta_1 - \Delta)|g_1\rangle\langle g_1| + \hbar(\Delta_2 - \Delta)|g_2\rangle\langle g_2| \\ &+ \frac{\hbar}{2}\left(\Omega_1\sigma_1^\dagger + \Omega_1^*\sigma_1\right) + \frac{\hbar}{2}\left(\Omega_2\sigma_2^\dagger + \Omega_2^*\sigma_1\right)\,, \end{aligned} \tag{5.8}$$

where we applied an additional energy shift of $-\Delta$ to all levels, with $\Delta = (\Delta_1 + \Delta_2)/2$.

(a) **(0.5 points)** For the ansatz $|\psi(t)\rangle = \psi_1(t)\,|g_1\rangle + \psi_2(t)\,|g_2\rangle + \psi_e(t)\,|e\rangle$, write down the time-dependent Schrödinger equation.

(b) **(2 points)** Assuming that $\Delta \gg |\Delta_1 - \Delta_2|$ we can make an adiabatic approximation $\partial_t\psi_e \approx 0$ to obtain an effective two level system. Find the resonance condition for the effective two-level system and the effective Raman Rabi frequency Ω_R, i.e. the oscillating frequency of the effective two-level system at resonance.

6 Schrödingers Cat

10 points *Markus Aichhorn*, TU Graz

The superposition principle states that if $|\phi_a\rangle$ and $|\phi_b\rangle$ are two possible states of a quantum system, then also $(|\phi_a\rangle + |\phi_b\rangle)/\sqrt{2}$ is a possible state. We call this superposition. When applied to macroscopic systems, however, this leads to paradoxical situations. One of the most famous of these paradoxa is Schrödingers cat, where the cat is in a superposition of dead and alive. We will investigate whether such a state is in practice detectable.

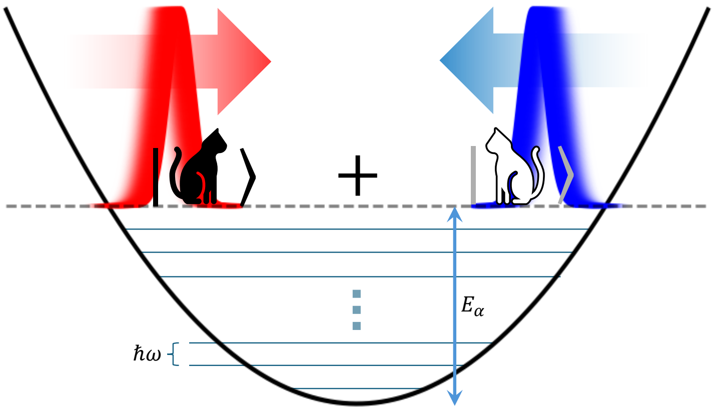


Figure 6.1: Schematic illustration of a cat state in the harmonic oscillator.

The basis of this example is the harmonic oscillator, which can realize macroscopic states, as we will see below. It is defined as

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 \tag{6.1}$$

As it is well known, this problem can be solved by introducing ladder operators. We define $\hat{X} = \hat{x}\sqrt{m\omega/\hbar}$ and $\hat{P} = \hat{p}/\sqrt{m\omega\hbar}$, and with that

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}), \quad \hat{N} = \hat{a}^\dagger\hat{a} \tag{6.2}$$

(a) **(1 point)** Check that if one works with functions of the dimensionless variables X and P, one has

$$\hat{P} = -i\frac{\partial}{\partial X}, \qquad \hat{X} = i\frac{\partial}{\partial P}$$

From the relation $\hat{a}\,|0\rangle = 0$, and replacing $\hat{a}$ by $\hat{X}$ and $\hat{P}$, calculate the wave function of the ground state $\psi_0(X)$ and $\psi_0(P)$. Do not normalize the result.

A special state is the eigenstate of the annihilation operator $\hat{a}$, i.e. $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. This *coherent state* has quasi-classical properties. It can be expanded in terms of $|n\rangle$, the eigenstates of the number operator $\hat{N}$, as follows

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{6.3}$$

An important property of $|\alpha\rangle$ is that it fulfills $\Delta x = \sqrt{\hbar/(2m\omega)}$ and $\Delta p = \sqrt{m\hbar\omega/2}$, i.e. $\Delta x \Delta p = \hbar/2$.

(b) **(1 point)** Following the same procedure as above, calculate the wave function $\psi_\alpha(X)$ of the coherent state $|\alpha\rangle$, as well as $\psi_\alpha(P)$. Again, normalisation is not necessary.

(c) **(2.5 points)** Suppose we have the system in a coherent state $|\alpha_0\rangle$ with $\alpha_0 = \rho e^{i\phi}$ at time $t = 0$. Show that the system is at later times in a coherent state which can be written as $e^{-i\omega t/2}|\alpha(t)\rangle$. Calculate $\langle x\rangle_t$ and $\langle p\rangle_t$. Now take $|\alpha| \gg 1$, look at $\Delta x/\langle x\rangle_{\max}$ and $\Delta p/\langle p\rangle_{\max}$, and argue why this can be called a quasi-classical state.

(d) **(0.5 points)** Consider a classical numerical example: We take now an ideal pendulum of length 1 meter and weight 1 gram. At time $t = 0$ the pendulum is at rest at $\langle x\rangle_{\max} = 1\,\mu$m. What is the corresponding value of $\alpha(t = 0)$ and the relative uncertainty on its position $\Delta x/\langle x\rangle_{\max}$?

We now define a *cat state* as follows,

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|\alpha\rangle + e^{i\pi/4}|-\alpha\rangle\right) \tag{6.4}$$

(e) **(0.5 points)** Suppose now α purely imaginary, i.e. $\alpha = i\rho$. Discuss qualitatively the physical properties of the composition of state (6.4) (position, momentum). For a value $|\alpha| \gg 1$, in what sense can this state be considered a concrete realization of the Schrödinger cat type of state?

We now study the properties of the cat state (6.4) in exactly this limit, $|\alpha| \gg 1$ with $\alpha = i\rho$, and we set $p_0 = \rho\sqrt{m\hbar\omega/2}$.

(f) **(2 points)** Calculate the (non-normalized) probability distribution for the position and the momentum of the system.

(g) **(0.5 points)** A physicist (Alice) prepares N independent systems all in the state (6.4) and measures the momentum of each of these systems. The measuring apparatus has a resolution δp which fulfills $\Delta p = \sqrt{m\hbar\omega/2} \ll \delta p \ll p_0$. Draw qualitatively the histogram of the results of the N measurements for $N \gg 1$.

(h) **(0.5 points)** Another physicist (Bob) claims that the measurements of Alice have not been done on a superposition of states as in (6.4), but on a non-paradoxical statistical mixture of states, that is to say half of the N systems are in state $|\alpha\rangle$, and the other half in $|-\alpha\rangle$. Assuming this is true, does one obtain the same probability distribution for the momentum as for the previous question for the N measurements?

(i) **(1 point)** In order to settle the matter, Alice now measures the position of the N independent systems, all prepared in state (6.4). Draw the shape of the resulting distribution of measured events, assuming that the resolution of the measuring apparatus is $\delta x \ll \frac{\sqrt{\hbar/2m\omega}}{\rho}$. Can Bob obtain the same result concerning the N position measurements assuming he is dealing with a statistical mixture?

(j) **(0.5 points)** Considering the numerical value obtained in the case of a simple pendulum in question 4), evaluate the resolution δx which is necessary in order to tell the difference between a set of N systems in a superposition (6.4) from a statistical mixture of states? Can this be done in practice?

7 Boltzmann Machine

8 points *Johannes Krondorfer*, TU Graz

The 2024 Nobel Prize in Physics recognized John J. Hopfield and Geoffrey Hinton for their foundational work on neural networks and machine learning — including the development of the Boltzmann Machine. This model applies principles from statistical mechanics to machine learning: configurations have energies, and learning adjusts these energies to match observed data. Boltzmann Machines introduced core ideas that shaped modern machine learning — energy-based models, expectation matching, and deep generative architectures. Their quantum extensions now aim to model entangled or coherent systems. In this problem, you will explore how Boltzmann Machines perform learning as free energy minimization, both in the classical and quantum setting, to approximate the distribution of given data.

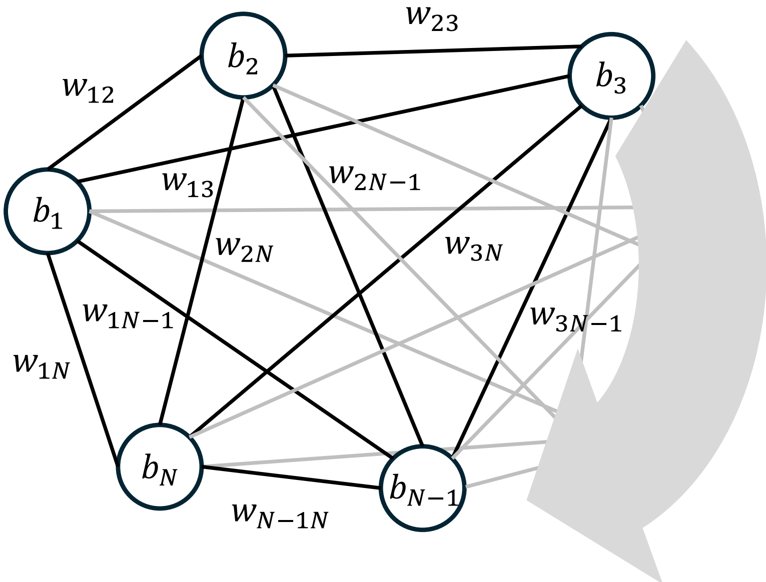


Figure 7.1: Illustration of a Boltzmann machine as an undirected graph, with edge weights w_{ij} and node weights b_i.

7.1 Classical Boltzmann Machine

A Boltzmann Machine is a stochastic neural network inspired by statistical physics. It consists of binary variables $s_i \in \{-1, 1\}$, representing the state of unit i, connected by symmetric interactions, as illustrated in Figure 7.1. The system defines a probability distribution over configurations based on an energy function

$$E(\boldsymbol{s}) = -\sum_{i<j} w_{ij} s_i s_j - \sum_i b_i s_i \,, \tag{7.1}$$

with $\boldsymbol{s} = (s_1, s_2, \ldots, s_N) \in \{-1, 1\}^N$, symmetric $w_{ij} \in \mathbb{R}$, and $b_i \in \mathbb{R}$.

(a) **(0.5 points)** Define the probability of a state $\boldsymbol{s}$ as

$$P(\boldsymbol{s}) = \frac{1}{Z} \mathrm{e}^{-\beta E(\boldsymbol{s})} \,. \tag{7.2}$$

Determine Z appropriately and compute the information $-\log P(\boldsymbol{s})$.

We define the entropy S as expected information, i.e.

$$S := -\sum_{\boldsymbol{s}\in\{-1,1\}^N} P(\boldsymbol{s}) \log P(\boldsymbol{s}) \,, \tag{7.3}$$

and the free energy as

$$F := -\frac{1}{\beta} \log Z \,. \tag{7.4}$$

(b) **(0.5 points)** Show that $F = \langle E \rangle_P - \frac{1}{\beta} S$, where $\langle . \rangle_P$ is the expectation with respect to P.

In the context of machine learning the thus defined system is used to model the distribution of given data P_{data}. The parameters of the model (w_{ij}, b_i) should be adjusted such that $P \approx P_{\text{data}}$. For that purpose we want to minimize the so-called Kullback-Leibler (KL) divergence

$$D(P_{\text{data}}||P) := \sum_{\boldsymbol{s}} P_{\text{data}}(\boldsymbol{s}) \log \frac{P_{\text{data}}(\boldsymbol{s})}{P(\boldsymbol{s})} \,, \tag{7.5}$$

which measures the deviation of the probability distributions P_{data} and P.

(c) **(1 point)** Show that $D(P_{\text{data}}||P) \geqslant 0$ and $D(P_{\text{data}}||P) = 0$ if and only if $P_{\text{data}} = P$.

(d) **(0.5 points)** Show that minimization of the KL-divergence is equivalent to maximization of the log-likelihood $\langle \log P \rangle_{P_{\text{data}}}$.

(e) **(1.5 points)** In order to optimize the parameters of the model, we can perform gradient descent. For that purpose compute explicitly the update rule for the parameters $\theta = w_{ij}, b_i$

$$\theta \leftarrow \theta - \eta \frac{\partial}{\partial \theta} D(P_{\text{data}}||P) \,, \tag{7.6}$$

with learning rate η. Interpret the result, what does the learning procedure correspond to?

In practice, Boltzmann Machines are used not just to learn distributions, but to generate samples $\boldsymbol{s} \sim P \approx P_{\text{data}}$ after training. However, since $P(\boldsymbol{s})$ cannot be computed exactly for large systems due to the intractability of the partition function Z, we rely on approximate sampling techniques. The most commonly used is Gibbs sampling, a Markov chain Monte Carlo (MCMC) method where variables are updated sequentially according to their conditional probabilities $P(s_i|\boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i}$ denotes $\boldsymbol{s}$ without s_i. Over time, the chain converges toward the target distribution $P(s)$.

(f) **(1 point)** Compute the conditional probability

$$P(s_i = 1|\boldsymbol{s}_{-i}) = \frac{1}{1 + \mathrm{e}^{-\alpha x}} \,, \tag{7.7}$$

by explicitly identifying α and x in terms of the model parameters and the current state. This expression is known as the *sigmoid function*, widely used in computer science.

7.2 Quantum Boltzmann Machine

In a Quantum Boltzmann Machine, the classical binary variables $s_i \in \{-1, 1\}$ are replaced by a quantum two-level system $\{|-1\rangle, |1\rangle\}$ as computational basis, and the energy function is generalized to a Hamiltonian H acting on a Hilbert space. The state of the system is described by a density matrix

$$\rho = \frac{1}{Z}\mathrm{e}^{-\beta H}, \qquad Z = \mathrm{tr}\{\mathrm{e}^{-\beta H}\}, \tag{7.8}$$

where $\mathrm{tr}\{.\} = \sum_n \langle n| \,.\, |n\rangle$ for some basis $|n\rangle$ of the underlying Hilbert space. This defines a quantum generalization of the Boltzmann distribution, where H is a Hermitian operator, possibly non-diagonal in the computational basis.

The principle from above remains the same and we define entropy

$$S := -\mathrm{tr}\{\rho \log \rho\}, \tag{7.9}$$

and free energy

$$F := -\frac{1}{\beta}\log Z. \tag{7.10}$$

(a) **(0.5 points)** Show that $F = \mathrm{tr}\{\rho H\} - \frac{1}{\beta}S$.

Analogously to the classical case we define the quantum Kullback-Leibler energy between a data density operator ρ_{data} and the model density operator ρ as

$$D(\rho_{\mathrm{data}}\|\rho) = \mathrm{tr}\left\{\rho_{\mathrm{data}}(\log \rho_{\mathrm{data}} - \log \rho)\right\}, \tag{7.11}$$

for which the same properties hold as in the classical case.

(b) **(1 point)** Assume we have a Hamiltonian of the form $H = \sum_\mu \theta_\mu O_\mu$, where O_μ are hermitian observables. Compute explicitly the update rule for the parameters

$$\theta_\mu \leftarrow \theta_\mu - \eta \frac{\partial}{\partial \theta_\mu} D(\rho_{\mathrm{data}}||\rho), \tag{7.12}$$

and compare the result with the classical one.

(c) **(1.5 points)** Consider a Hamiltonian of the form

$$H = -\sum_{i<j} w_{ij}\sigma_i^z\sigma_j^z - \sum_i b_i\sigma_i^z, \tag{7.13}$$

with symmetric $w_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$ and σ_i^z the z-Pauli matrix at position i.[14] Argue why this system in the computational basis $\{|-1\rangle, |1\rangle\}$ for each state is not more powerful than the classical analog and provide an extension that takes the quantum nature into account.

[14]The z-Pauli matrix is given by $\sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where the i-th position refers to the i-th "particle". More explicitly this means that $\sigma_i^z = \overbrace{I \otimes I \otimes \cdots \otimes I}^{i-1} \otimes \sigma^z \otimes \overbrace{I \cdots \otimes I}^{N-i}$.

8 The Precession of Mercury

A Practical Approach to Geodesics in General Relativity

14 points

Matthias Diez, TU Graz & KFU

In the formalism of general relativity it can be quite tedious to deal with all the indices. In this problem, we want to take an easier approach to get the equations of motions of a massive particle when a specific metric is given. We explicitely want to calculate the fraction of perihelion precession of mercury that may be accounted for by general relativity. The line element in general relativity is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

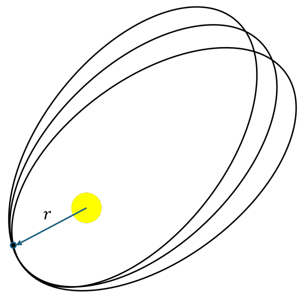


Figure 8.1: Perihelion precession

8.1 The Geodesic Equation

In order to derive the geodesic equation, usually we try to minimize the action $\int_{x_1}^{x_2} ds$

$$0 = \delta \int_{x_1}^{x_2} ds = \delta \int_{x_1}^{x_2} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \delta \int_{u_1}^{u_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}} du \,, \tag{8.1}$$

where the final equality holds if the curve is parametrized with u.

(a) **(1 point)** Derive from the variation of ds the Geodesic equation of motion and write it as:

$$\frac{d^2}{du^2} x^\mu - \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{du} \frac{dx^\mu}{du} = 0 \,, \tag{8.2}$$

with the Christoffel-Symbol,

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2} g^{\mu\lambda} [\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\rho\nu}]. \tag{8.3}$$

Hereby it is useful to use the normalization condition of the four-velocity for a massive particle, $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1$, what would the normalization condition for a Photon be?

(b) **(1 point)** Show, using the Euler-Lagrange equations , that the Lagrangians $L = \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}$ and $\mathcal{L} = L^2$ lead to the same equations of motion.

8.2 The Schwarzschild Metric

Next we will consider the well known Schwarzschild metric, which is a good description for a "2 body-problem" in our solar system. This metric can be written as (units: G = 1, c=1)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{8.4}$$

(a) **(0.5 points)** Identify the Lagrangian $\mathcal{L}$ in equation (8.4).

(b) **(1 point)** Identify the constants of motion of $\mathcal{L}$.

(c) **(1 point)** Sketch how you would determine the Christoffel Symbols by comparing the equations of motion directly with equation (S8.4), without using expression (8.3) and explicitly calculate in this way one of the Christoffel symbols.

8.3 Classical Treatment

Before we try to discuss the relativistic case any further, let us look at Newtonian orbits. For this we consider to be in a plane, and use polar coordinates.

(a) **(0.5 points)** Write down the non-relativistic kinetic energy of a particle in polar coordinates.

(b) **(0.5 points)** Write down the Lagrangian of a particle moving in one plane of a central potential, generated by a body with mass M and identify the constants of motion.

(c) **(0.5 points)** Express, $\dot{r}^2$ in terms of r and the constants of motion (Hint replace $\dot{\phi}$ with an expression related to angular momentum).

(d) **(1 point)** In a next step we introduce a new variable $p := \frac{1}{r}$ and replace the derivatives with respect to t with respect to $\dot{\phi}$. This should give;

$$(p'(\phi))^2 + p(\phi)^2 = Ap(\phi) + B \tag{8.5}$$

Determine the constants A and B.

(e) **(1 point)** Now take the derivative on both sides of this equations with respect to ϕ and solve the differential equation for p.

8.4 Relativistic Treatment

Let us now use the constants of motion we defined before

(a) **(1.5 points)** Use $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1$ the constants of motion from $\mathcal{L}$, and $\theta = \frac{\pi}{2}$ to express $\dot{r}^2$.

(b) **(0.5 points)** In a next step we introduce again $p := \frac{1}{r}$, and replace the derivatives with respect to s with respect to ϕ.

(c) **(1 point)** Derive both sides of this equation with respect to ϕ and you should get:

$$p''(\phi) + p(\phi) = C + Dp(\phi)^2 \tag{8.6}$$

Determine the constants C and D.

(d) **(1 points)** Justify why the term Dp^2 can be seen as a small perturbation for Mercury.

(e) **(2 points)** Use the Newtonian solution from equation (8.5) in the additional Term Dp^2 and write down the differential equation for the first order perturbation $p^{(1)}$. Use the particular solution y_p of

$$y''(x) + y(x) = A(1 + 2B + B\cos(x)), \quad y_p = A\left(1 + Bx\cos(x) + B\left[\frac{1}{2} - \frac{1}{6}\cos(x)\right]\right). \tag{8.7}$$

and determine an expression for the precession of mercury, (i.e. how far the focci rotate for one revolution). (Hint: The approximation $\cos(\phi - \beta) = \cos\phi\cos\beta + \sin\phi\sin\beta$ might be useful.

9 Pulsar Electrodynamics

10 points *Martin Napetschnig*, TU Munich

[5] Pulsars are among the most extreme objects in our universe. Pulsars are rapidly spinning neutron stars (NS). These are compact objects (size of $\sim \mathcal{O}(10\text{ km})$), which are formed at the end of a stars lifecycle, if the stars mass is about $1.2 - 3.6\, M_\odot$. As you will show soon, neutron stars are fastly rotating (can be as fast as milliseconds!), are permeated with a very high magnetic field and send *pulsating* radiation around their rotation axis, that we can detect on Earth and on Earth bound satellites. Pulsars are used as reliable 'clocks' in astronomy. In fact, in 2023 an array of so-called *pulsar timing arrays (PTAs)* has been used to detect a stochastic nHz gravitational wave background for the first time. Neutron stars are an active field of research and there is still a lot to explore. In this exercise, you will analytically derive some of the characteristic properties of pulsars.

If you feel more comfortable using Gauss (cgs) units instead of SI, there will be no points deducted for wrong factors of μ_0 & c.

9.1 Pulsar Characteristics

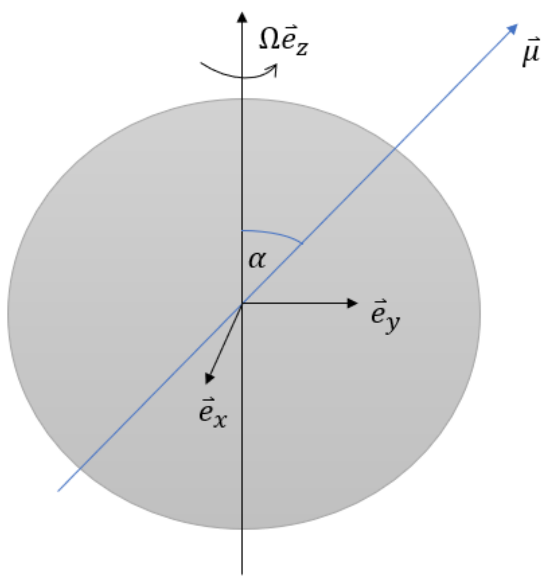


Figure 9.1: Sketch of the pulsar. The magnetic moment $\vec{\mu}$ is tilted from the rotation axis by a constant angle α.

(a) **(0.5 points)** Given a star with total mass M, angular rotation velocity Ω and radius R, calculate the minimal period that it can have by equating the centrifugal to the gravitational force (on the equator). Insert numbers for our Sun and a typical neutron star and give an answer about whether their measured periods are close to their minimum period or not.

(b) **(0.5 points)** During a stellar collapse to a NS, the mass of the star remains constant, while it's radius shrinks drastically. Using that angular momentum and magnetic flux are conserved, find the relation between the initial and final rotation velocity $\frac{\Omega_f}{\Omega_i}$ and the initial and final magnetic field $\frac{B_f}{B_i}$ as a function of $\frac{R_i}{R_f}$. How much would the magnetic field of our Sun be enhanced if it kept its mass and would shrink to a NS? (Our Sun is actually too light for this, but if it was twice as heavy it could do so).

(c) **(2 points)** We take the rotation axis of the pulsar to be the z-axis. The magnetic field far away from the pulsar is modeled as a *dipole* field with a definite magnetic moment $\vec{\mu}$, which is pointing in a direction tilted by a fixed angle α from the z-axis and rotating with the pulsar (see fig. 9.1).
The pulsar is loosing a lot of energy via radiation, which will cause it to **spin down**. Put differently, the energy for the emission of radiation needs to be supplied by the kinetic energy of the pulsar. The radiated power is given by the dipole radiation formula:

$$\mathcal{P} = -\frac{\mu_0}{12\pi c^3}\left|\ddot{\vec{\mu}}\right|^2 \tag{9.1}$$

Calculate the kinetic energy loss per time $\frac{dE}{dt}$ for a rigid full sphere rotating with constant mass and radius. Equate your result to (9.1) and find a relation between the period $P = \frac{2\pi}{\Omega}$ and it's rate of change $\dot{P}$. [15]

(d) **(1 point)** Astronomers observe pulsars and can measure the present values $\dot{P}_0$ & P_0 and purely from this information infer the age of the pulsar T. Show that they are no better than you and provide a formula $T(P_0, \dot{P}_0)$. [16]

Useful constants: $R_\odot \sim 7\times10^8\,\mathrm{m}$, $M_\odot \sim 1.9\times10^{30}\,\mathrm{kg}$, $P_\odot \sim 2.3\times10^6\,\mathrm{s}$ (27 days), $B_\odot(R_\odot) = 10^{-4}\,\mathrm{T}$, $G_N = 6.67\times10^{-11}\frac{\mathrm{m}^3}{\mathrm{kg s}^2}$, $R_{\mathrm{NS}} \sim 10^4\,\mathrm{m}$, $M_{\mathrm{NS}} \sim M_\odot$, $P_{\mathrm{NS}} \sim 10^{-3} - 1\,\mathrm{s}$, $B_{\mathrm{NS}}(R_{\mathrm{NS}}) = 10^4 - 10^{11}\,\mathrm{T}$.

9.2 The Aligned Rotator

To study the magnetosphere of the pulsar, we now specialize to the case where the internal magnetic field inside the pulsar is aligned with the rotation axis, i.e. $\vec{B}_{in}||\vec{\Omega}||\vec{\mu}$. This model is called the *aligned rotator*[17]. As of (9.1), in this case there is no dipole radiation emitted, but as you are going to show now, there is an inevitable outflow of particles that will cause an additional source of radiation. The pulsar is modeled as a rigid, conducting full sphere that has an internal magnetic field $\vec{B}_{in} = B_0\vec{e}_z$. The region outside the sphere is vacuum ($\rho_{out} = \vec{j}_{out} = 0$). I recommend to use standard spherical coordinates, where the angle θ is defined as the angle α before.

[15] *Hint*: Parametrize $\vec{\mu} = \mu\,\vec{e}_\mu(t)$ with $|\vec{e}_\mu|^2 = 1$ and insert it into (9.1).
[16] *Hint*: Assume the period has increased by a lot, i.e. $P_0 >> P_{\mathrm{initial}}$.
[17] The non-aligned rotator has been solved in a very thorough calculation by Armin J. Deutsch in 1955

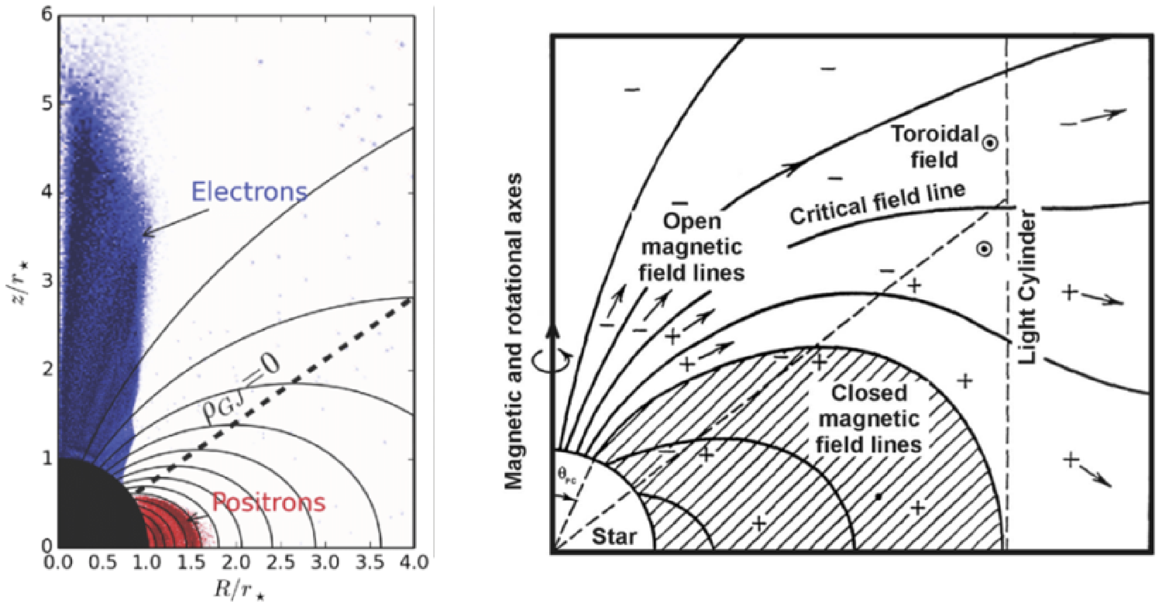


Figure 9.2: *Left:* Computer simulations of the charge distribution around the pulsar. Magnetic field lines are black, the GJ boundary line that you calculate in point (c) is dashed. Taken from *Crinquand 2021*.
Right: Similar to the left plot. Open and closed field lines are shown together with θ_{PC} that you derive in point (d). The GJ boundary is again the dashed line. Moreover, the light cylinder distance R_L is also shown. Taken from *Venter, 2008*. In these sources $\Omega = -\Omega \vec{e}_z$, such that the charges are opposite to our case.

(a) **(2 points)** Outside the pulsar, the magnetic field $\vec{B}_{out}$ is a dipole field. The corresponding vector potential reads

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\times\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\frac{\vec{\mu}}{r} \tag{9.2}$$

Calculate $\vec{B}_{out}$ from (9.2) and show that it is given by

$$\vec{B}_{out} = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\,r^2 - 3(\vec{\mu}\cdot\vec{r})\,\vec{r}}{r^5} \tag{9.3}$$

In case you can not derive this, proceed with (9.3) for the next points. Give its components in the basis vectors of spherical coordinates using the vector identities stated below: $\vec{B}_{out} = B_r\vec{e}_r + B_\theta\vec{e}_\theta + B_\phi\vec{e}_\phi$. By matching the $\vec{B}$ field inside and outside the star at the polar caps ($\theta = 0, \pm\pi$), find the relation between μ and B_0.

(b) **(1 point)** Mathematically, field lines are curves $r(\theta)$ satisfying

$$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} \tag{9.4}$$

Find $r(\theta)$ up to an integration constant K that we will use later. Each value of K 'labels' a field line. This result is generic for dipole fields and gives them the 'toroidal' shape (see also fig. 9.2).

(c) **(2 points)** Our initial assumption was that the pulsar is in vacuum. However, upon calculating the electric field strength on the surface (which we will not do), one finds that the electric forces are much stronger than gravity, such that charged particles are extracted of the pulsar and form a *corotating magnetosphere* consisting of a fully conducting plasma that rotates with the pulsar. Find the charge density $\rho(r,\theta)$ using that in a fully conducting plasma the Lorentz force has to vanish in combination with Maxwells equations/Gauss law ($\vec{\nabla}\cdot\vec{E} = \ldots$). You should recover a famous result first found by Goldreich & Julian in 1969:

$$\rho_{GJ} = -2\varepsilon_0(\vec{\Omega}\cdot\vec{B}) \tag{9.5}$$

For which values of θ we have positive/negative charges in the magnetosphere? At which value of θ is the boundary line between the two regimes? [18]

(d) **(1 point)** It is not surprising that our approximations are doomed to fail sooner or later. The assumption of a corotating plasma at larger and larger distances has to break down because physics dictates a fundamental speed limit to the orbital velocity of the plasma particles. What is the maximum distance R_L at which the orbital velocity reaches its maximally allowed upper value? This value is called the radius of the *light cylinder*. Magnetic field lines that bend back 'soon enough' to not touch the light cylinder are *closed field lines*. They bend and feed back into the star (see figure 9.2). Field lines which start at a too low angle do not bend before reaching the light cylinder, their fate is beyond our theoretical control. These field lines are called *open field lines*. They can be an additional source of radiation and/or charged particles. Find the critical angle at the polar cap θ_{PC}, below which one has open field lines. To do so, use your result for the field line $r(\theta)$ that you derived before.

One may need the following conversions between unit vectors in Cartesian and spherical coordinates:

$$\begin{pmatrix}\vec{e}_x\\ \vec{e}_y\\ \vec{e}_z\end{pmatrix} = \begin{pmatrix}\sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi\\ \sin\theta\,\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}\vec{e}_r\\ \vec{e}_\theta\\ \vec{e}_\phi\end{pmatrix} \tag{9.6}$$

This is a short reminder of some useful vector identities:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A}, \tag{9.7}$$
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}), \tag{9.8}$$
$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C}), \tag{9.9}$$
$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - (\vec{A}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{A}) \tag{9.10}$$
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}), \tag{9.11}$$
$$\Delta\frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta^{(3)}(\vec{r}-\vec{r}'), \tag{9.12}$$
$$\varepsilon_{ijk}\varepsilon^{ijk} = 3! = 6, \tag{9.13}$$
$$\vec{e}_r\times\vec{e}_\theta = \vec{e}_\phi \tag{9.14}$$
$$\vec{e}_\theta\times\vec{e}_\phi = \vec{e}_r \tag{9.15}$$
$$\vec{e}_\phi\times\vec{e}_r = \vec{e}_\theta. \tag{9.16}$$

[18] *Hint*: The electric field is stationary, i.e. does not depend on time and there are no currents outside the pulsar, $\vec{j}_{out} = 0$. Moreover, (9.11) might be useful.

Solutions

S1 Problem Quartet

12 points

Matthias Diez, TU Graz & KFU

S1.1 Bouncing Balls

(a) Momentum and energy conservation give us:

$$m_1 v - m_2 v = m_1 v_1 + m_2 v_2 \tag{S1.1}$$

$$m_1 \frac{v^2}{2} + m_2 \frac{v^2}{2} = m_1 \frac{v_1^2}{2} + m_2 \frac{v_2^2}{2} \tag{S1.2}$$

combining these two equations leads to

$$v + v_1 = v_2 - v \tag{S1.3}$$

and from this we end up after reinsertion at:

$$v_1 = \frac{m_1 - 3m_2}{m_1 + m_2} v = \frac{1 - 3a}{1 + a} v \tag{S1.4}$$

$$v_2 = \frac{3m_1 - m_2}{m_1 + m_2} v = \frac{3 - a}{1 + a} v \tag{S1.5}$$

where we used a $= m_2/m_1$.

(b) The maximum velocity v_2 is reached for $a \to 0$, and this gives $v_2 = 3v$

(c) If $a = \frac{1}{3}$ ball 1 comes to rest according to equation (S1.5) and from this we directly get $v_2 = 2v$

(d) We have seen that if ball 1 bounces off ball 2 and is much lighter it gains exactly v. In the co-moving system of ball 1, ball two moves with velocity -2v before the collision and is reflected with velocity 2v after the collision. Thus the lab frame ball two moves with velocity $v_2 = 3v$. The third ball then moves with velocity -4_v in the comoving frame of ball 2 before the collision. It is then reflected with 4v, and in the lab frame we get $v_5 = 4v + 3v = 7v$

S1.2 Falling Conductor Loop

From height h to l the conductor is freely falling, this means it arrives with velocity $v_z(t_0) = -\sqrt{2g(h-l)}at$ at the top of the non zero magnetic field area at a time $t_0 = \sqrt{2(h-l)/g}$. After entering due to Faraday's law the induced voltage is

$$U = -\frac{d\phi}{dt} = -Bw\frac{dz}{dt} \tag{S1.6}$$

Therefrom the induced current is:

$$I = \frac{U}{R} \tag{S1.7}$$

and this induced current leads to a Lorentz force on the loop which is:

$$F = -\frac{Bw^2 v_z}{R}\mathbf{e_z} \tag{S1.8}$$

The equation of motion for the loop is then:

$$m\frac{dv_z}{dt} = -mg - \frac{B^2 w^2 v_z}{R} \tag{S1.9}$$

We can solve this equation by separation of variables:

$$-\int_{v_z(t_0)}^{v_z(t)} \frac{dv_z'}{g + \alpha v_z'} = \int_{t_0}^{t} dt'. \tag{S1.10}$$

with $\alpha = \frac{B^2 w^2}{mR}$ Solving the integrals on both sides and rearranging gives:

$$v_z(t) = -\frac{g}{\alpha}\left[1 - e^{-\alpha(t-t_0)}\right] + v_z(t_0)e^{-\alpha(t-t_0)} \tag{S1.11}$$

S1.3 Fata Morgana

Snell's law gives

$$n(Y)\sin\theta_0 = n(z)\sin(\theta(z))\,. \tag{S1.12}$$

Furthermore we can use the trigonometric identity

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\cot^2\theta}}\,, \tag{S1.13}$$

and $\cot\theta = \frac{dz}{dx}$ to arrive at a differential equation

$$\frac{dz}{dx} = \sqrt{\frac{n(z)^2}{n(Y)^2\sin\theta_0^2} - 1}\,, \tag{S1.14}$$

describing the trajectory of the light beam. As the problem is symmetric we find the conditions that at $x = 0$ the derivative of z vanishes and $z(x=0) > H$. We can then solve equation (S1.14) by separation of variables:

$$\int_Y^z dz' \frac{1}{\sqrt{\frac{(1+Az')^2}{(1+Ay)^2\sin\theta_0^2} - 1}} = \int_{-D}^{x} dx' \tag{S1.15}$$

This integral can be solved by substituting $\cosh(u) = \frac{1+Az}{1+Ay\sin\theta_0}$, leading to

$$z(x) = \frac{1}{A}\left[(1+Ay\sin\theta_0)\cosh\left(\frac{(x+D)A}{1+Ay\sin\theta_0} + \cosh^{-1}\left[\frac{1}{\sin\theta_0}\right]\right) - 1\right]. \tag{S1.16}$$

The condition that $\frac{dz}{dx}(x=0)$ vanishes gives

$$A = \frac{\sin\theta_0 - 1}{z(0) - y\sin\theta_0}\,. \tag{S1.17}$$

With the help of a computer one can then try to solve the last two equations for A.

S1.4 Water Reservoir

We use the bernoulli equation with:

$$\rho\left(\frac{1}{2}v_1^2 + gh(t)\right) = \frac{1}{2}\rho v_A^2 \tag{S1.18}$$

When we assume that r is much smaller than the radius of the water surface, we may say that $v_1 \ll v_A$ and can therfore be neglected. Thus the velocity of the water leaving the reservoir is

$$v_A = \sqrt{2gh(t)} \tag{S1.19}$$

The current water volume in the reservoir is

$$V = \frac{1}{3}R(h(t))^2\pi h(t) = \frac{1}{3}k^2\pi h^3(t) \tag{S1.20}$$

The change in volume is

$$\frac{dV}{dt} = k^2\pi h^2\frac{dh}{dt} \tag{S1.21}$$

This is equivalent to the amount of water flowing out, which is:

$$I = -r^2\pi v_B = -r^2\pi\sqrt{2gh(t)} \tag{S1.22}$$

We then get by separation of variables:

$$\int_{h_0}^{h(t)} h'^{\frac{3}{2}}dh' = -\int_0^t \frac{r^2}{k^2}\sqrt{2g}dt' \tag{S1.23}$$

which results in.

$$h = h_0\left(1 - \frac{5r_0^2\sqrt{2g}}{2k^2h_0^{\frac{5}{2}}}t\right)^{\frac{2}{5}} \tag{S1.24}$$

The velocity at a distance d is

$$v_d(t) = \sqrt{v_A(t)^2 + 2gd} = \sqrt{2g(h(t) + d}. \tag{S1.25}$$

We use the continuity equation:

$$r_d(t)^2\pi v_d(t) = r^2 v_A(t) \tag{S1.26}$$

Therefrom we have:

$$r_d(t) = r\sqrt{\frac{v_A(t)}{v_d(t)}} = r\left(1 + \frac{d}{h(t)}\right)^{-\frac{1}{4}} \tag{S1.27}$$

Plugging in (S1.24) we end up with:

$$r_d(t) = \left(1 + \frac{d}{h_0}\left(1 - \frac{5r_0^2\sqrt{2g}}{2k^2h_0^{\frac{5}{2}}}t\right)^{\frac{2}{5}}\right)^{-\frac{1}{4}} \tag{S1.28}$$

S2 Kapitza's Pendulum

13 points

Johannes Krondorfer, TU Graz

S2.1 Equations of Motion

(a) We use the angle θ as generalized coordinate and write $x = \ell\sin(\theta)$ and $y = y_0 - \ell\cos(\theta)$. With that we get

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m\ell^2}{2}\dot{\theta}^2 \\ V &= mg(y_0 - \ell\cos(\theta)) \,. \end{aligned} \tag{S2.1}$$

By using the Lagrange function $L = T - V$ and the Euler-Lagarange equation $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$, we get

$$m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = 0\,, \tag{S2.2}$$

and thus

$$\ddot{\theta} = -\frac{g}{\ell}\sin(\theta) \tag{S2.3}$$

(b) The stationary points are easily found by computing $\ddot{\theta} = 0$ which is solved by $\theta = n\pi$ with $n \in \mathbb{Z}$. By linearization we perform a stability analysis of $\theta = 0$ and $\theta = \pi$.

$\underline{\theta = 0}$: For small angles $\sin(\theta) \approx \theta$ by Taylor expansion. Thus, we get the linearized equation

$$\ddot{\theta} = -\frac{g}{\ell}\theta\,, \tag{S2.4}$$

which has $\sin(\omega_0)$ and $\cos(\omega_0)$ solutions, with $\omega_0 = \sqrt{\frac{g}{\ell}}$. So $\theta = 0$ is stable.

$\underline{\theta = \pi}$: By setting $\phi = \theta - \pi$ we get $\sin(\phi + \pi) \approx -\phi$. Thus we have

$$\ddot{\phi} = \frac{g}{\ell}\phi\,, \tag{S2.5}$$

which has real exponential solutions and is thus unstable.

(c) Now we use the time dependent pivot point $y_0(t)$. The equations remain the same, but one has to take the time dependence of $y_0(t)$ into account. This yields

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\ell^2\cos(\theta)^2\dot{\theta}^2 + (\dot{y}_0 + \ell\sin(\theta)\dot{\theta})^2) \\ &= \frac{m}{2}(\ell^2\cos(\theta)^2\dot{\theta}^2 + \ell^2\sin(\theta)^2\dot{\theta}^2 + 2\dot{y}_0\ell\sin(\theta)\dot{\theta} + \dot{y}_0^2) \\ &= \frac{m\ell^2}{2}\dot{\theta}^2 + m\ell\dot{y}_0\sin(\theta)\dot{\theta} + \frac{m}{2}\dot{y}_0^2 \end{aligned} \tag{S2.6}$$

where the term not depending on θ or $\dot{\theta}$ is irrelevant. The potential V is the same as before, but with time dependent $y_0(t)$. For the Euler Lagrange equation we thus get

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} \\ &= \Big(m\ell^2\ddot{\theta} + m\ell(\ddot{y}_0\sin(\theta) + \dot{y}_0\cos(\theta)\dot{\theta})\Big) - \Big(m\ell\dot{y}_0\cos(\theta)\dot{\theta} - mg\ell\sin(\theta)\Big) \\ &= m\ell^2\ddot{\theta} + m\ell(\ddot{y}_0 + g)\sin(\theta)\,. \end{aligned} \tag{S2.7}$$

Thus we can write

$$\ddot{\theta} = -\frac{\ddot{y}_0 + g}{\ell}\sin(\theta)\,. \tag{S2.8}$$

The additional appearing term $\ddot{y}_0$ clearly acts as an additional acceleration due to the forced movement of the pendulum.

(d) Similar to the linearization before we can write $\sin(\theta) \approx \theta$ for $\theta \ll 1$ and $\sin(\phi + \pi) \approx -\phi$ for $\phi \ll 1$ and $\theta = \phi + \pi$. Thus, by defining $x_0 = \theta, \phi$ and $x_1 = \dot{x}_0 = \dot{\theta}, \dot{\phi}$ we get the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ \mp\frac{\ddot{y}_0+g}{\ell} & 0 \end{bmatrix}\boldsymbol{x}\,, \tag{S2.9}$$

where the $+$ is for the expansion around 0 and the $-$ is for the expansion around π. So we have $\alpha_\pm(t) = \mp\frac{\ddot{y}_0(t)+g}{\ell}$.

S2.2 Floquet-Lyapunov Theorem

(a) To prove the Floquet-Lyapunov theorem we first prove the identity $U(t+T) = U(t)U(T)$. For that we define $V(t) = U(t+T)U^{-1}(T)$ and compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \frac{\mathrm{d}}{\mathrm{d}t}U(t+T)U^{-1}(T) = H(t+T)U(t+T)U^{-1}(T) = H(t)V(t)\,, \tag{S2.10}$$

and $V(0) = U(T)U^{-1}(T) = I$, and thus $V(t) = U(t)$ as it satisfies the same defining differential equation. Now in order to show the theorem we define $P(t) = U(t)\mathrm{e}^{-\widetilde{H}t}$ and show P is T-periodic by

$$\begin{aligned} P(t+T) &= U(t+T)\mathrm{e}^{-\widetilde{H}(t+T)} = U(t)U(T)\mathrm{e}^{-\widetilde{H}T}\mathrm{e}^{-\widetilde{H}t} \\ &= U(t)U(T)U^{-1}(T)\mathrm{e}^{-\widetilde{H}t} = U(t)\mathrm{e}^{-\widetilde{H}t} = P(t)\,, \end{aligned}$$

where we have used $\mathrm{e}^{-\widetilde{H}T} = \mathrm{e}^{-\log U(T)} = U^{-1}(T)$. This concludes the proof of the theorem.

(b) First we apply the transformation and write $U = PW$, and thus

$$\begin{aligned} HPW = HU = \dot{U} &= \dot{(PW)} = \dot{P}W + P\dot{W} \\ &= HPW - P\widetilde{H}W + P\dot{W}\,, \\ \Rightarrow \dot{W} &= \widetilde{H}W \end{aligned}$$

where we used $\dot{P} = (U\mathrm{e}^{-\widetilde{H}t})^{\cdot} = \dot{U}\mathrm{e}^{-\widetilde{H}t} - U\mathrm{e}^{-\widetilde{H}t}\widetilde{H} = HP - P\widetilde{H}$. The differential equation for W has constant coefficients, so standard stability arguments apply: $W(t)$ is stable iff the spectrum of $\widetilde{H}$ is a subset of the left half plane, i.e. $\sigma(\widetilde{H}) \subseteq \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \leqslant 0\}$, because then $||W|| \leqslant C$ for some constant C. For the stability of U the situation does not change, since P is periodic, invertible and continuous we can estimate

$$\begin{aligned} ||U(t)|| &= ||P(t)W(t)|| \leqslant ||P(t)||\,||W(t)|| \leqslant \max_{\tau\in[0,T]} ||P(\tau)||\,||W(t)|| \\ ||W(t)|| &= ||P(t)^{-1}U(t)|| \leqslant ||P(t)^{-1}||\,||U(t)|| \\ &\Rightarrow ||U(t)|| \geqslant ||P(t)^{-1}||^{-1}||W(t)|| \geqslant \min_{\tau\in[0,T]} ||P(\tau)^{-1}||^{-1}||W(t)||\,. \end{aligned}$$

And thus we have

$$C'||W|| \leqslant ||U|| \leqslant C''||W|| \tag{S2.11}$$

which means that the stability of U is determined by W and thus by $\widetilde{H}$.

S2.3 Dyson Series and Magnus Expansion

(a) We insert the ansatz (2.7) into the ODE (2.4) and gather terms of the same order in H. This yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} U^{(k)}(t) &= \sum_{k=0}^{\infty} H U^{(k)}(t) \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} U^{(k)}(t) &= H U^{(k-1)}(t)\,, \text{ for } k \geqslant 1\,. \end{aligned}$$

So we get

$$\begin{aligned} U^{(1)}(t) &= \int_0^t H(t_1)\,\mathrm{d}t_1 \\ U^{(k)}(t) &= \int_0^t U^{(k-1)}(t_1)\,\mathrm{d}t_1\,, \end{aligned}$$

and thus we directly obtain (2.8).

(b) In principle we can just start with (2.8) and integrate every time $t_1, \dots, t_k$ from 0 to t. However, we have to correct multiple appearing terms by dividing with $k!$, which is the number of possible permutations of the $H(t_i)$. With that we can write

$$U^{(k)}(t) = \frac{1}{k!}\left(\int_0^t H(t')\,\mathrm{d}t'\right)^k\,, \tag{S2.12}$$

and thus

$$U(t) = \sum_{k=0}^{\infty} U^{(k)}(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_0^t H(t')\,\mathrm{d}t'\right)^k = \exp\left(\int_0^t H(t')\,\mathrm{d}t'\right)\,. \tag{S2.13}$$

Alternatively one can just show that the exponential ansatz satisfies the differential equation (2.4). For our linearized pendulum, however, this is not the case since the system matrix does not commute for different times, instead we have

$$\left[\begin{bmatrix} 0 & 1 \\ \alpha_\pm(t) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \alpha_\pm(t') & 0 \end{bmatrix}\right] = (\alpha_\pm(t') - \alpha_\pm(t)) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,.$$

(c) To compute the first few terms of the Magnus expansion it is easiest to expand on both sides of the equation

$$\begin{aligned} \exp(\Omega(t)) &= U(t) \\ \Leftrightarrow \quad \exp\Big(\sum_{k=1}^{\infty} \Omega^{(k)}(t)\Big) &= \sum_{k=0}^{\infty} U^{(k)}(t) \\ \Leftrightarrow I + \Omega^1 + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \mathcal{O}(||H||^4) &= I + U^{(1)} + U^{(2)} + U^{(3)} + \mathcal{O}(||H||^4) \\ \Leftrightarrow \begin{matrix} \Omega^{(1)} + \\ \Omega^{(2)} + \frac{1}{2}(\Omega^{(1)})^2 \\ \Omega^{(3)} + \frac{1}{2}(\Omega^{(1)}\Omega^{(2)} + \Omega^{(2)}\Omega^{(1)}) + \frac{1}{6}(\Omega^{(1)})^3 + \\ \mathcal{O}(||H||^4) \end{matrix} &= \begin{matrix} U^{(1)} + \\ U^{(2)} + \\ U^{(3)} + \\ \mathcal{O}(||H||^4) \end{matrix}\,. \end{aligned}$$

Now simple rearrangement yields the result (2.10).

S2.4 Stability Analysis of Kapitza's Pendulum

(a) To compute $\widetilde{H}_\pm$ we compute the individual terms, noting that $\Omega_\pm^{(2)}(T) = 0$. We have

$$\frac{\Omega_\pm^{(1)}}{T} = \frac{1}{T}\int_0^T \begin{bmatrix} 0 & 1 \\ \alpha_\pm(t') & 0 \end{bmatrix} \mathrm{d}t' = \begin{bmatrix} 0 & 1 \\ \mp\omega_0^2 & 0 \end{bmatrix},$$

where we used

$$\frac{1}{T}\int_0^T \ddot{y}_0(t')\,\mathrm{d}t' = 0\,.$$

So we only have $U^{(3)}(T)$ left to calculate, which is given by

$$\begin{aligned} U^{(3)}(T) &= \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \begin{bmatrix} 0 & 1 \\ \alpha_\pm(t_1) & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \alpha_\pm(t_2) & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \alpha_\pm(t_3) & 0 \end{bmatrix} \\ &= \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \begin{bmatrix} 0 & \alpha_\pm(t_2) \\ \alpha_\pm(t_3)\alpha_\pm(t_1) & 0 \end{bmatrix}. \end{aligned}$$

And we have

$$\begin{aligned} \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3\; \alpha_\pm(t_2) &= \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\; t_2\alpha_\pm(t_2) \\ &= \mp \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\; t_2 \left(\frac{A\omega^2}{\ell}\cos(\omega t_2) + \omega_0^2\right) \\ &= \mp\frac{A\omega^2}{\ell}\int_0^T \mathrm{d}t_1 \underbrace{\int_0^{t_1} \mathrm{d}t_2\; t_2\cos(\omega t_2)}_{\frac{t_1\sin(\omega t_1)}{\omega} + \frac{\cos(\omega t_1)}{\omega^2} - \frac{1}{\omega^2}} \mp \frac{\omega_0^2 T^3}{6} \\ &= \mp\frac{\omega_0^2 T^3}{6} \pm \frac{2AT}{\ell}, \end{aligned}$$

where we used that $\int_0^T \mathrm{d}t_1 \frac{t_1\sin(\omega t_1)}{\omega} = -\frac{T}{\omega^2}$. For the other term we get

$$\begin{aligned} &\int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3\; \alpha_\pm(t_1)\alpha_\pm(t_3) \\ &= \int_0^T \mathrm{d}t_1 \left(\mp\frac{A\omega^2}{\ell}\cos(\omega t_1) \mp \omega_0^2\right) \overbrace{\int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \left(\mp\frac{A\omega^2}{\ell}\cos(\omega t_3) \mp \omega_0^2\right)}^{\pm\frac{A}{\ell}\cos(\omega t_1) \mp \frac{A}{\ell} \mp \frac{\omega_0^2 t_1^2}{2}} \\ &= -\frac{A^2\omega^2 T}{2\ell^2} + \frac{2A\omega_0^2 T}{\ell} + \frac{\omega_0^4 T^3}{6}, \end{aligned}$$

With that we have everything together and calculate

$$\begin{aligned} \Omega^{(3)} &= U^{(3)} - \frac{1}{6}(\Omega^{(1)})^3 \\ &= \begin{bmatrix} 0 & \mp\frac{\omega_0^2T^3}{6} \pm \frac{2AT}{\ell} \\ -\frac{A^2\omega^2 T}{2\ell^2} + \frac{2A\omega_0^2 T}{\ell} + \frac{\omega_0^4T^3}{6} & 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 0 & T \\ \mp\omega_0^2 T & 0 \end{bmatrix}^3 \\ &= \begin{bmatrix} 0 & \mp\frac{\omega_0^2T^3}{6} \pm \frac{2AT}{\ell} \\ -\frac{A^2\omega^2 T}{2\ell^2} + \frac{2A\omega_0^2 T}{\ell} + \frac{\omega_0^4T^3}{6} & 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 0 & \mp\omega_0^2 T^3 \\ \omega_0^4 T^3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \pm\frac{2AT}{\ell} \\ -\frac{A^2\omega^2 T}{2\ell^2} + \frac{2A\omega_0^2 T}{\ell} & 0 \end{bmatrix}. \end{aligned}$$

In total we get

$$\begin{aligned}\widetilde{H}_\pm &\approx \frac{\Omega^{(1)\pm} + \Omega^{(3)}_\pm}{T} \\ &= \begin{bmatrix} 0 & 1 \pm \frac{2A}{\ell} \\ \mp\omega_0^2 - \frac{A^2\omega^2}{2\ell^2} + \frac{2A\omega_0^2}{\ell} & 0 \end{bmatrix} \\ &\approx \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}\left(\frac{A\omega}{\ell}\right)^2 \mp \omega_0^2 & 0 \end{bmatrix}.\end{aligned}$$

(b) Transforming the equation back into a second order differential equation we get

$$\ddot{\theta}_\pm = \left(-\frac{1}{2}\left(\frac{A\omega}{\ell}\right)^2 \mp \omega_0^2\right)\theta_\pm\,. \tag{S2.14}$$

The solution is stable if $-\frac{1}{2}\left(\frac{A\omega}{\ell}\right)^2 \mp \omega_0^2 < 0$, which is the case if $\left(\frac{A\omega}{\ell}\right)^2 > \mp 2\omega_0^2$. This is always the case for the initially stable point $\theta_0 = 0$. For $\theta_0 = \pi$ this can be achieved by suitably adjusting A and ω. This yields new frequencies of the pendulum as

$$\omega_\pm^2 = \frac{1}{2}\left(\frac{A\omega}{\ell}\right)^2 \pm \omega_0^2 \qquad \text{if} \qquad \left(\frac{A\omega}{\ell}\right)^2 > \mp 2\omega_0^2 \tag{S2.15}$$

S3 Relativistic Particle in a Box

10 points *Martin Napetschnig*, TU Munich

(a) Apart from the irrelevant constant mc^2, the Lagrange function is

$$L(x,\dot{x}) \doteq \frac{m\dot{x}^2}{2} - V(x) + \mathcal{O}\left(\left(\frac{\dot{x}}{c}\right)^4\right) \tag{S3.1}$$

This leads to $m\ddot{x} = -\partial_x V(x)$, Newton's second law.

(b)

$$p = \frac{\partial L}{\partial \dot{x}} = \gamma m \dot{x} = m\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}, \tag{S3.2}$$

$$\frac{p^2}{m^2} = \dot{x}^2\left(1+\frac{p^2}{m^2c^2}\right), \tag{S3.3}$$

$$\dot{x}(p) = \frac{c\,p}{\sqrt{m^2c^2+p^2}} \tag{S3.4}$$

Then

$$H(p,x) = p\,\dot{x}(p) - L(x,\dot{x}(p)), \tag{S3.5}$$

$$= \frac{c^2p^2}{\sqrt{m^2c^4+c^2p^2}} + \frac{m^2c^4}{\sqrt{m^2c^4+c^2p^2}} + V(x), \tag{S3.6}$$

$$= \sqrt{c^2p^2+m^2c^4} + V(x) \tag{S3.7}$$

For $p = 0 = V(x)$, we recover $H = E = mc^2$, Einsteins formula.

(c) Not much to show here: $(H-V)\Psi = \sqrt{\ldots}\Psi \Rightarrow (H-V)^2\Psi = \ldots\Psi$

(d) One can easily see that (3.3) solves (3.2) if $k = \pm\frac{1}{c}\sqrt{(E-qV)^2 - m^2c^4}$

(e) We calculate first the group velocity for all cases (using $E = \sqrt{c^2p^2+m^2c^4} + qV_0$):

$$v_G := \frac{\partial E}{\partial p} = \frac{c^2\,p}{\sqrt{c^2p^2+m^2c^4}} = \frac{c^2\,p}{E-qV_0} \tag{S3.8}$$

Now, we see that:

- Weak potential: Here, A_{II} corresponds to a right-moving and B_{II} to a left-moving state, as we are used to. This is also the case for A_{I} and B_{I}
- Intermediate potential: In these cases, k is imaginary and A_{II} is an exponentially decaying state while B_{II} would be exponentially growing and is thus unphysical.
- Strong potential: Chapeau who gets this right! As $E - qV_0 < 0$, the roles of A_{II} and B_{II} are reversed! A_{II} is **left**-moving and B_{II} is **right**-moving.

(f) For this problem I can put the wall at the origin($x = 0$). The first derivative of Ψ can have a kink, but must be continuous. Thus the procedure is valid. We know already that $r = B_{\mathrm{I}}$. From the previous point we also derived that $A_{\mathrm{II}} = 0$, since it describes left-moving states. This leaves us with $t = B_{\mathrm{II}}$. From the matching of Ψ and Ψ' one gets:

$$1 + r = t, \tag{S3.9}$$

$$k(1 - r) = -k't \tag{S3.10}$$

with $k = \frac{1}{c}\sqrt{E^2 - m^2c^4}$ and $k' = \frac{1}{c}\sqrt{(E - qV_0)^2 - m^2c^4}$. One then finds

$$r = \frac{k + k'}{k - k'}, \tag{S3.11}$$

$$t = \frac{2k}{k - k'} \tag{S3.12}$$

This is all for this point. It should be pointed out that if the roles of $A_{\mathrm{II}} = 0 \leftrightarrow B_{\mathrm{II}} = 0$ are exchanged, $k' \leftrightarrow -k'$ and one treats the case of the weak potential regime, which is completely analogous to the usual thing we do in the QM lectures.

(g) Both k and k' are real numbers, so

$$|t|^2 = t^2 = \frac{4k^2}{(k - k')^2}, \tag{S3.13}$$

$$R = |r|^2 = \frac{(k + k')^2}{(k - k')^2}, \tag{S3.14}$$

$$T = 1 - R = -4\frac{kk'}{(k - k')^2}, \tag{S3.15}$$

$$T < 0 \tag{S3.16}$$

$$R > 1 \tag{S3.17}$$

The very last line is the essence of the Klein paradox. We seem to get more stuff reflected than we sent in. Moreover, less than zero is transmitted, which is also counter-intuitive. In case of the weak potential regime, $k' \leftrightarrow -k'$ and we obtain the regular results that we are familiar with: $0 < R, T < 1$ & $R + T = 1$. In the intermediate potential regime, $k' = i\kappa$ is complex and $R = 1, T = 0$. The fact that $T \neq |t|^2$ is a generic result and is not specific to our example. What is conserved is the *probability flux* $v|\Psi(t,x)|^2 \sim \frac{k}{m}|\Psi(t,x)|^2$. For the reflection coefficient $R = \frac{k_R}{k_I}|r|^2 = 1|r|^2$, where $k_I = k = k_R$, while for the transmission coefficient $T = \frac{k_T}{k_I}|t|^2 = \frac{\pm k'}{k}|t|^2$. The sign of k' now depends on whether one is in the weak or in the strong potential regime. In the weak one $k' > 0$, $0 < T < 1$, while in the strong one $k' < 0$, $T < 0$. As an aside, conservation of probability is ensured in all cases via $R + T = |r|^2 + \frac{\pm k'}{k}|t|^2 = 1$. We see in (S3.14) that for $V_0 \to \infty$, $\lim_{k' \to \infty} R = 1$. This is what we wanted: The infinite square well is fully reflective, **but it does not reflect more than what is incident**.

The resolution of the apparent paradox is that in region II, the wavefunction is actually describing a *negative* energy solution, i.e. antiparticles. Solid State physicists might prefer to think about it as a reversed biased **pn-junction** (Zener diode) where V_0 would be the built-in voltage. The potential barrier is large enough to source particle-antiparticle/electron-hole pairs ($qV_0 > mc^2$). **Antiparticles** have negative charge and feel a potential *dip* instead of a *well*. This causes a negative charge current to the right, equivalent to a positive charge current to the left, explaining the negative transmission coefficient. **Particles** are repelled by the potential and get pushed to the left, adding up with the fully reflected incident beam, thus accounting for $R > 1$.

(h) The eigenfunctions are completely analogous to the non-relativistic case.

$$\Psi(t,x) = A\sin(kx)e^{-\frac{i}{\hbar}Et}\,. \tag{S3.18}$$

The cos solution does not allow $\Psi(t, x = 0) = 0$. Since $\Psi(t, x = L) = 0 \to k = \frac{n\pi}{L}, n \in \mathbb{N}$. For the normalization we need to solve

$$|A|^2 \int_0^L dx\ \sin^2(k\,x) = 1 = \tag{S3.19}$$

$$|A|^2 \left(\frac{L}{2} - \frac{1}{2k}\cos(kx)\sin(kx)|_{x=0}^{x=L}\right) \Rightarrow \tag{S3.20}$$

$$A = \sqrt{\frac{2}{L}}, \Rightarrow \tag{S3.21}$$

$$\Psi(t,x) = \sqrt{\frac{2}{L}}\sin\left(n\pi\frac{x}{L}\right)e^{-\frac{i}{\hbar}E_n t} \tag{S3.22}$$

where E_n are obtained by plugging (S3.22) into (3.2) with the result:

$$E_n = mc^2\sqrt{1 - \frac{\hbar^2\pi^2 n^2}{m^2c^2L^2}} \tag{S3.23}$$

The suitable expansion parameter is the *Compton wavelength* of the particle (versus the system size L): $\lambda_C = \frac{\hbar}{mc}$. We can Taylor expand (S3.23) and find:

$$E_n = mc^2 + \frac{\hbar^2\pi^2 n^2}{2mL^2} + \mathcal{O}\left(\left(\frac{\lambda_C}{L}\right)^4\right) \tag{S3.24}$$

As long as the box size is much larger than the Compton wavelength, the non-relativistic result is an excellent approximation.

S4 Ion Trap Chips

15 points

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S4.1 Macroscopic Paul Traps

(a) Consider an electrostatic field $\varphi(\vec{x})$ in free-space. It must satisfy the Laplace equation $\Delta\varphi(\vec{x}_0) = 0$ at all points $\vec{x}_0$.
In terms of the Hessian matrix $(H\varphi)_{i,j}(\vec{x}_0) = (\partial_i\partial_j\varphi)(\vec{x}_0)$, we can write the Laplace equation as $\mathrm{tr}H\varphi(\vec{x}_0) = 0$. But since the trace is the sum of the eigenvalues of a matrix, this means that the Hessian of the potential φ has either at least one negative eigenvalue or all eigenvalues are zero at the point $\vec{x}_0$.
Hence, the potential φ cannot have a minimum at $\vec{x}_0$ and, therefore, one cannot stably confine an electrically charged particle in this potential.

(b) Electric potential of an infinitely long cylinder of radius R with line charge λ:

$$\varphi(r) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r}{R}\right) \tag{S4.1}$$

Hence, for the shown configuration:

$$\begin{aligned}\varphi(x,y,t) = \frac{\lambda(t)}{2\pi\varepsilon_0}\Bigg(&-\ln\left(\frac{\sqrt{(d-x)^2+y^2}}{R}\right) - \ln\left(\frac{\sqrt{(d+x)^2+y^2}}{R}\right) \\ &+ \ln\left(\frac{\sqrt{x^2+(d-y)^2}}{R}\right) + \ln\left(\frac{\sqrt{x^2+(d+y)^2}}{R}\right)\Bigg).\end{aligned} \tag{S4.2}$$

Therefore:

$$\varphi(x,y,t) = \frac{\lambda(t)}{4\pi\varepsilon_0} \ln\left(\frac{(x^2+(d-y)^2)(x^2+(d+y)^2)}{((d-x)^2+y^2)((d+x)^2+y^2)}\right). \tag{S4.3}$$

Taylor expansion around $x = 0, y = 0$ up to second order:

$$\begin{aligned}\varphi(x,y,t) = \varphi(0,0,t) &+ \left.\frac{\partial\varphi}{\partial x}\right|_{x=0,y=0} x + \left.\frac{\partial\varphi}{\partial y}\right|_{x=0,y=0} y + \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{x=0,y=0} xy \\ &+ \frac{1}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{x=0,y=0} x^2 + \frac{1}{2}\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{x=0,y=0} y^2.\end{aligned} \tag{S4.4}$$

Evaluating the terms:

$$\varphi(0,0,t) = 0, \tag{S4.5}$$

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=0,y=0} = 0, \quad \left.\frac{\partial\varphi}{\partial y}\right|_{x=0,y=0} = 0, \tag{S4.6}$$

$$\left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{x=0,y=0} = 0, \quad \left.\frac{\partial^2\varphi}{\partial x^2}\right|_{x=0,y=0} = \frac{8}{d^2}, \quad \left.\frac{\partial^2\varphi}{\partial y^2}\right|_{x=0,y=0} = -\frac{8}{d^2}. \tag{S4.7}$$

Putting this together, we obtain:

$$\varphi(x,y,t) = \frac{\lambda(t)}{\pi\varepsilon_0}(x^2-y^2) = \frac{\lambda_0\cos\Omega t}{\pi\varepsilon_0}(x^2-y^2). \tag{S4.8}$$

(c) The equations of motion are given by:

$$m\ddot{x} = -e\frac{\partial\varphi}{\partial x} = -\frac{2\lambda_0 e\cos\Omega t}{\pi\varepsilon_0 d^2}x, \tag{S4.9}$$

and

$$m\ddot{y} = -e\frac{\partial\varphi}{\partial y} = \frac{2\lambda_0 e\cos\Omega t}{\pi\varepsilon_0 d^2}y. \tag{S4.10}$$

For $\Omega = 0$, the equations of motion become:

$$m\ddot{x} = -\frac{2\lambda_0 e}{\pi\varepsilon_0 d^2}x,\ \ m\ddot{y} = \frac{2\lambda_0 e}{\pi\varepsilon_0 d^2}y. \tag{S4.11}$$

We see that a charged particle will harmonically oscillate in the x-direction, but will be unconfined in the y-direction.

(d) Using the given equation for the effective potential ψ:

$$\psi(x,y) = \frac{e}{4m\Omega^2}|\vec{\nabla}\varphi(x,y)|^2 = \frac{e}{m\Omega^2}\frac{\lambda_0^2}{\pi^2\varepsilon_0^2}\left(x^2+y^2\right). \tag{S4.12}$$

The classical Hamiltonian for the motion of the ion in the trap is hence given by:

$$H = \frac{p_x^2+p_y^2}{2m} + e\psi(x,y) = \frac{p_x^2+p_y^2}{2m} + \frac{e^2}{m\Omega^2}\frac{\lambda_0^2}{\pi^2\varepsilon_0^2}\left(x^2+y^2\right) = \frac{p_x^2+p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2), \tag{S4.13}$$

with

$$\omega = \sqrt{2}\frac{e}{m\Omega}\frac{\lambda_0}{\pi\varepsilon_0}. \tag{S4.14}$$

Quantizing $x = \sqrt{\hbar/2m\omega}(a_x^\dagger + a_x)$, $p_x = i\sqrt{\hbar m\omega/2}(a_x^\dagger - a_x)$ and $y = \sqrt{\hbar/2m\omega}(a_y^\dagger + a_y)$, $p_y = i\sqrt{\hbar m\omega/2}(a_y^\dagger - a_y)$, we obtain:

$$H = \hbar\omega\left(a_x^\dagger a_x + \frac{1}{2}\right) + \hbar\omega\left(a_y^\dagger a_y + \frac{1}{2}\right). \tag{S4.15}$$

Assuming two internal states $|g\rangle, |e\rangle$ of energies E_g, E_e, we obtain for the quantum-mechanical Hamiltonian of the system:

$$H = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| + \hbar\omega\left(a_x^\dagger a_x + \frac{1}{2}\right) + \hbar\omega\left(a_y^\dagger a_y + \frac{1}{2}\right), \tag{S4.16}$$

where the oscillation frequencies in x- and y-direction are equally $\omega_x = \omega_y = \omega = \sqrt{2}\frac{e}{m\Omega}\frac{\lambda_0}{\pi\varepsilon_0}$.

S4.2 Micro-fabricated Ion Traps

(a) We want to solve the electrostatic problem:

$$\begin{cases} \Delta\varphi = 0 & \text{in } \mathbb{H}_{\geqslant 0}, \\ \varphi(x,z,y) = f(x,z) & \text{for } y = 0 \end{cases} \tag{S4.17}$$

in $\mathbb{H}_{\geqslant 0} = \{(x,z,y)|y \geqslant 0\}$ with

$$f(x,z) = \begin{cases} V & \text{for } x_1 \leqslant x \leqslant x_2, \\ 0 & \text{otherwise} \end{cases}. \tag{S4.18}$$

There are different ways to solve it. We will use Green's function. The Green function in $\mathbb{H}_{\geqslant 0}$ is, by the method of mirror charges, given by:

$$G(x,z,y,x',z',y') = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2+(z-z')^2+(y-y')^2}} - \frac{1}{\sqrt{(x-x')^2+(z-z')^2+(y+y')^2}}\right) \tag{S4.19}$$

Using Green's theorem in the form

$$\int_{\partial V}\left(\varphi\vec{\nabla}G - G\vec{\nabla}\varphi\right)\cdot \mathrm{d}\vec{S} = \int_V (\varphi\Delta G - G\Delta\varphi)\,\mathrm{d}^3r, \tag{S4.20}$$

we obtain with the property $\Delta G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')$:

$$\varphi(\vec{r}) = \frac{1}{\varepsilon_0}\int_V \mathrm{d}^3r'\rho(\vec{r}')G(\vec{r},\vec{r}') + \int_{\partial V} f(\vec{r}')\vec{\nabla}_{r'}G(\vec{r},\vec{r}')\cdot\mathrm{d}\vec{S}_{r'}, \tag{S4.21}$$

which becomes for $\rho(\vec{r}) = 0$ in $\mathbb{H}_{>0}$:

$$\varphi(x,z,y) = \int_{x_1}^{x_2}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}z f(x,z)\frac{\partial}{\partial y'}\bigg|_{y'=0} G(x,z,y,x',z',y'). \tag{S4.22}$$

Plugging in:

$$\frac{\partial}{\partial y'}\bigg|_{y'=0} G(x,z,y,x',z',y') = \frac{1}{2\pi}\frac{y}{\left((x-x')^2+(z-z')^2+y^2\right)^{3/2}} \tag{S4.23}$$

and hence:

$$\varphi(x,z,y) = \frac{V}{2\pi}\int_{x_1}^{x_2}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}z'\frac{1}{\left((x-x')^2+(z-z')^2+y^2\right)^{3/2}}, \tag{S4.24}$$

$$\varphi(x,z,y) = \frac{V}{\pi}\int_{x_1}^{x_2}\mathrm{d}x'\frac{y}{(x-x')^2+y^2}. \tag{S4.25}$$

Finally:

$$\varphi(x,z,y) = \varphi(x,y) = \frac{V}{\pi}\left(\arctan\left(\frac{x_2-x}{y}\right) - \arctan\left(\frac{x_1-x}{y}\right)\right). \tag{S4.26}$$

Alternative Solution:

We consider Laplace's equation in two dimensions:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{S4.27}$$

Given boundary conditions at $y = 0$:

$$\phi(x,0) = V(x), \tag{S4.28}$$

where $V(x)$ is a prescribed function (e.g., a strip potential).

The Fourier transform of $\phi(x,y)$ is defined as:

$$\tilde{\phi}(k, y) = \int_{-\infty}^{\infty} \phi(x, y) e^{-ikx} \, dx. \tag{S4.29}$$

Applying the Fourier transform to Laplace's equation:

$$\mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = 0. \tag{S4.30}$$

Since differentiation in x corresponds to multiplication by $-k^2$ in Fourier space:

$$(-k^2)\tilde{\phi}(k, y) + \frac{\partial^2 \tilde{\phi}}{\partial y^2} = 0. \tag{S4.31}$$

Rearranging:

$$\frac{\partial^2 \tilde{\phi}}{\partial y^2} - k^2 \tilde{\phi} = 0. \tag{S4.32}$$

This is a standard second-order differential equation with the general solution:

$$\tilde{\phi}(k, y) = A(k) e^{-|k| y}. \tag{S4.33}$$

The term $e^{|k|y}$ is excluded to ensure the solution does not diverge as $y \to \infty$.
Using the boundary condition $\tilde{\phi}(k, 0) = \tilde{V}(k)$, we get:

$$\tilde{\phi}(k, y) = \tilde{V}(k) e^{-|k| y}. \tag{S4.34}$$

To obtain $\phi(x, y)$ in real space:

$$\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(k) e^{-|k| y} e^{ikx} \, dk. \tag{S4.35}$$

For a conducting strip of length $L = x_2 - x_1$, the boundary condition is:

$$V(x) = \begin{cases} V, & x_1 \leqslant x \leqslant x_2, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform of this function is:

$$\tilde{V}(k) = V \int_{x_1}^{x_2} e^{-ikx} \, dx = V \frac{e^{-ikx_1} - e^{-ikx_2}}{-ik}. \tag{S4.36}$$

Substituting into the inverse Fourier integral:

$$\phi(x, y) = \frac{V}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx_1} - e^{-ikx_2}}{-ik} e^{-|k| y} e^{ikx} \, dk. \tag{S4.37}$$

This integral was given leading to:

$$\phi(x,y) = \frac{V}{\pi}\left[\tan^{-1}\left(\frac{x-x_1}{y}\right) - \tan^{-1}\left(\frac{x-x_2}{y}\right)\right]. \tag{S4.38}$$

—

We now evaluate the integral:

$$I = \int_0^\infty \frac{\sin(ak)}{k} e^{-bk}\cos(ck)\,dk. \tag{S4.39}$$

Using Euler's formula:

$$\cos(ck) = \frac{e^{ick}+e^{-ick}}{2}, \tag{S4.40}$$

we write:

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(ak)}{k} e^{-bk}(e^{ick}+e^{-ick})\,dk. \tag{S4.41}$$

Splitting:

$$I = \frac{1}{2}\left[I_1 + I_2\right], \tag{S4.42}$$

where

$$I_1 = \int_0^\infty \frac{\sin(ak)}{k} e^{-(b-ic)k}\,dk, \quad I_2 = \int_0^\infty \frac{\sin(ak)}{k} e^{-(b+ic)k}\,dk. \tag{S4.43}$$

Using $\sin(ak) = \frac{e^{iak}-e^{-iak}}{2i}$:

$$I_1 = \frac{1}{2i}\int_0^\infty \frac{e^{i(a+c)k}-e^{i(a-c)k}}{k} e^{-bk}\,dk. \tag{S4.44}$$

A standard integral result is:

$$\int_0^\infty \frac{e^{(i\alpha-b)k}}{k}\,dk = -\ln(b-i\alpha). \tag{S4.45}$$

Applying this,

$$I = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{c+a}{b}\right) - \tan^{-1}\left(\frac{c-a}{b}\right)\right]. \tag{S4.46}$$

—

(b) Using the result from the previous exercise, we obtain:

$$\varphi(t,x,y) = \frac{V_0}{\pi}\cos(\Omega t)\left(\arctan\left(\frac{a+b-x}{y}\right) - \arctan\left(\frac{a-x}{y}\right) + \arctan\left(\frac{c+x}{y}\right) - \arctan\left(\frac{x}{y}\right)\right). \tag{S4.47}$$

(c) Because of the relation $\psi(x,y) = \frac{e}{4m\Omega^2}|\vec{\nabla}\varphi(x,y)|^2$ for the effective potential, the desired minimum (x_0, y_0) of the effective potential ψ is as well an extremum of the electric potential $\varphi(x,y)$, where $\vec{\nabla}\varphi(x_0,y_0) = 0$.
Calculating $\partial\varphi(x_0,y)/\partial y = 0$ for $b = c$ yields:

$$y_0 = \frac{1}{2}\sqrt{a(a+2b)}, \tag{S4.48}$$

i.e. the ion-surface separation depends on the electrode sizes. One can therefore change y_0 by changing the electrode sizes a, b.

(d) Taylor expansion of φ around (x_0, y_0):

$$\varphi(x,y) = \varphi(x_0,y_0) + \left.\frac{\partial\varphi}{\partial x}\right|_{(x_0,y_0)} x + \left.\frac{\partial\varphi}{\partial y}\right|_{(x_0,y_0)} y + \frac{1}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{(x_0,y_0)} x^2 + \frac{1}{2}\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{(x_0,y_0)} y^2 + \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{(x_0,y_0)} xy + \ldots \tag{S4.49}$$

Because of the form of $\varphi(x_0, y_0)$ given in the problem statement, it suffices to evaluate $\partial_x^2\varphi$ at $x_0 = a/2$ and $y_0 = \sqrt{a(a+2b)}/2$:

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{(x_0,y_0)} = \frac{V}{\pi}\frac{8b}{(a+b)^2\sqrt{a(a+2b)}} = \frac{V_0}{y_0^2}\frac{1}{4}a(a+2b)\frac{8b}{\pi(a+b)^2\sqrt{a(a+2b)}} = \frac{V_0}{y_0^2}\kappa, \tag{S4.50}$$

and thus:

$$\kappa = \frac{2b}{\pi(a+b)^2}\sqrt{a(a+2b)}. \tag{S4.51}$$

S4.3 Double-wells

(a) C.f. Brown, K., Ospelkaus, C., Colombe, Y. *et al.*: *Coupled quantized mechanical oscillators.* Nature **471**, 196–199 (2011), https://doi.org/10.1038/nature09721

The Coulomb interaction potential for the two trapped charged particles of charges Q_A, Q_B in the two potential wells separated by a distance s_0 is given by (+ Taylor expansion):

$$U(x_A, x_B) = \frac{1}{4\pi\varepsilon_0}\frac{Q_AQ_B}{s_0 - x_A + x_B} \approx \frac{Q_AQ_B}{4\pi\varepsilon_0 s_0}\left(1 + \frac{x_A - x_B}{s_0} + \frac{x_A^2}{s_0^2} + \frac{x_B^2}{s_0^2} - \frac{2x_Ax_B}{s_0^2}\right), \tag{S4.52}$$

where x_A and x_B are the displacements of the particles from the minima in the double-well potential.

The term proportional to x_Ax_B representes the lowest order coupling between the motions of the particles in the potential minima, hence:

$$U_c \approx -\frac{Q_AQ_B}{2\pi\varepsilon_0 s_0^3}x_Ax_B. \tag{S4.53}$$

The particles oscillate with harmonic frequencies ω_A, ω_B in their respective wells:

$$x_A = \sqrt{\frac{\hbar}{2m\omega_A}}\left(a_A^\dagger + a_A\right), \quad x_B = \sqrt{\frac{\hbar}{2m\omega_B}}\left(a_B^\dagger + a_B\right). \tag{S4.54}$$

Therefore:

$$\hat{U}_c = -\frac{Q_AQ_B}{4\pi\varepsilon_0 m s_0^3}\frac{\hbar}{\sqrt{\omega_A\omega_B}}\left(a_A^\dagger + a_A\right)\left(a_B^\dagger + a_B\right). \tag{S4.55}$$

Using the "rotating wave approximation" given in the problem statement, this becomes:

$$\hat{U}_c \approx\approx -\frac{Q_A Q_B}{4\pi\varepsilon_0 m s_0^3}\frac{\hbar}{\sqrt{\omega_A\omega_B}}\left(a_A^\dagger a_B + a_A a_B^\dagger\right) = -\hbar\Omega_{\text{ex}}\left(a_A^\dagger a_B + a_A a_B^\dagger\right). \tag{S4.56}$$

We therefore obtain for the coupling strength:

$$\hbar\Omega_{\text{ex}} = \hbar\frac{Q_A Q_B}{4\pi\varepsilon_0 m\sqrt{\omega_A\omega_B}s_0^3}. \tag{S4.57}$$

(b) The harmonic oscillations of the particles in their respective wells contribute each a term $\hbar\omega_i\left(a_i^\dagger a_i + 1/2\right)$ to the Hamiltonian. Together with the expression of the coupling strength between the particles in the double-well potential, derived in the previous sub-problem, this becomes:

$$H = \hbar\omega_A\left(a_A^\dagger a_A + \frac{1}{2}\right) + \hbar\omega_B\left(a_B^\dagger a_B + \frac{1}{2}\right) - \hbar\Omega_{\text{ex}}\left(a_A^\dagger a_B + a_A a_B^\dagger\right). \tag{S4.58}$$

(c) The Lagrangian of the system is given by:

$$L = \frac{1}{2}m\dot{x}_A^2 + \frac{1}{2}m\dot{x}_B^2 - \frac{1}{2}k_A x_A^2 - \frac{1}{2}k_B x_B^2 - \frac{1}{2}c\left(x_A - x_B\right)^2. \tag{S4.59}$$

Hence the equations of motion in matrix form:

$$\begin{pmatrix}\ddot{x}_A\\ \ddot{x}_B\end{pmatrix} = \begin{pmatrix}-\frac{k_A+c}{m} & \frac{c}{m}\\ \frac{c}{m} & -\frac{k_B+c}{m}\end{pmatrix}\begin{pmatrix}x_A\\ x_B\end{pmatrix}. \tag{S4.60}$$

The eigenfrequencies are given by the square-roots of the eigenvalues of the above matrix system to be:

$$\omega_\pm = \sqrt{\frac{1}{m}\left(\frac{k_A+k_B}{2} + c \pm \sqrt{\frac{\left(k_A-k_B\right)^2}{4} + c^2}\right)}. \tag{S4.61}$$

(d) In the case $k_A = k_B = k$, the eigenfrequencies of the system become:

$$\omega_+ = \sqrt{\frac{k+2c}{m}} \tag{S4.62}$$

and

$$\omega_- = \sqrt{\frac{k}{m}}, \tag{S4.63}$$

with the corresponding eigenvectors:

$$\vec{v}_+ = \begin{pmatrix}1\\ -1\end{pmatrix},\ \vec{v}_- = \begin{pmatrix}1\\ 1\end{pmatrix}. \tag{S4.64}$$

Therefore, the solution of the equation of motion is given by:

$$\vec{x}(t) = \begin{pmatrix}x_A(t)\\ x_B(t)\end{pmatrix} = C_+\vec{v}_+\cos\left(\omega_+ t + \phi_+\right) + C_-\vec{v}_-\cos\left(\omega_- t + \phi_-\right). \tag{S4.65}$$

Using the initial conditions, we obtain:

$$\vec{x}_A(t) = \frac{A_0}{2} \left(\cos\left(\omega_+ t\right) + \cos\left(\omega_- t\right)\right), \tag{S4.66}$$

and

$$\vec{x}_B(t) = \frac{A_0}{2} \left(\cos\left(\omega_- t\right) - \cos\left(\omega_+ t\right)\right). \tag{S4.67}$$

We want to know the smalles $t > 0$, for which:

$$\cos\left(\omega_+ t\right) + \cos\left(\omega_- t\right) = 0. \tag{S4.68}$$

We can solve this equation ($n \in \mathbb{N}$ odd):

$$\omega_+ t = \omega_- t + n\pi. \tag{S4.69}$$

Since we want the smalles $t > 0$ for which this is the case, we choose $n = 1$ and obtain:

$$t_{\text{ex}} = \frac{\pi}{\omega_+ - \omega_-}. \tag{S4.70}$$

S5 Hyperfine Qubits in Trapped Neutral Atoms

10 points

Johannes Krondorfer, TU Graz

S5.1 A Two Level Atom in a Laser Field

(a) The correct transformation rule for the Hamiltonian under a time-dependent unitary transformation

$$H' = VHV^\dagger + \mathrm{i}\hbar\dot{V}V^\dagger\,. \tag{S5.1}$$

Obviously the first term of the Hamiltonian commutes with the transformation and we have

$$i\hbar\dot{V}V = \mathrm{i}\hbar\cdot\mathrm{i}\omega\sigma^\dagger\sigma VV^\dagger = -\hbar\omega\sigma^\dagger\sigma\,,$$

and

$$\begin{aligned} V\sigma V^\dagger &= \begin{bmatrix} \mathrm{e}^{\mathrm{i}\omega t} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega t} & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{e}^{-\mathrm{i}\omega t}\sigma \\ V\sigma^\dagger V^\dagger &= \mathrm{e}^{\mathrm{i}\omega t}\sigma^\dagger\,. \end{aligned}$$

So in total we get

$$\begin{aligned} H' &= -\hbar\overbrace{(\omega-\omega_0)}^{\Delta}\sigma^\dagger\sigma + \hbar\left(-\overbrace{\frac{\boldsymbol{\mathcal{D}}\cdot\boldsymbol{E}_0}{2\hbar}}^{2\Omega}(\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}\omega t})\mathrm{e}^{\mathrm{i}\omega t}\sigma^\dagger - \frac{\boldsymbol{\mathcal{D}}^*\cdot\boldsymbol{E}_0}{2\hbar}(\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}\omega t})\mathrm{e}^{-\mathrm{i}\omega t}\sigma\right) \\ &\approx -\hbar\Delta\sigma^\dagger\sigma + \frac{\hbar}{2}\left(\Omega\sigma^\dagger + \Omega^*\sigma\right) \end{aligned}$$

(b) It is straight forward to rewrite the transformed Hamiltonian in terms of Pauli-matrices, by shifting the energy about $-\Delta/2$, which is always possible. Thus we have

$$H' = \hbar\frac{\Delta}{2}\mathbb{1} + \hbar\frac{\Delta}{2}\sigma_z + \hbar\frac{\Omega'}{2}\sigma_x - \hbar\frac{\Omega''}{2}\sigma_y\,,$$

with Ω', Ω'' the real and imaginary part of Ω. So we have

$$\begin{aligned} \alpha &= \hbar\frac{\Delta}{2} \\ \tilde{\boldsymbol{\Omega}} &= \begin{bmatrix}\Omega' & -\Omega'' & \Delta\end{bmatrix}^T \\ \tilde{\Omega} &= \sqrt{\Delta^2 + |\Omega|^2} \end{aligned}$$

(c) To prove this identity there are of course several ways. We will use Taylor expansion to prove it. Separating the odd and even parts of the Taylor expansion of the exponential we get

$$\begin{aligned} \exp(\mathrm{i}\theta\,\boldsymbol{n}\cdot\boldsymbol{\sigma}) &= \sum_{k=0}^{\infty}\frac{(\mathrm{i}\theta)^{2k}}{(2k)!}(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k} + \sum_{k=0}^{\infty}\frac{(\mathrm{i}\theta)^{2k+1}}{(2k+1)!}(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k+1} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!} + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \\ &= \cos(\theta)\mathbb{1} + \mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\theta)\,, \end{aligned}$$

where we used

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \sum_{i,j} n_i n_j \sigma_i \sigma_j = \frac{1}{2}\sum_{i,j} n_i n_j \{\sigma_i, \sigma_j\} = ||\boldsymbol{n}|| = 1\,.$$

(d) Now it is easy to calculate the time evolution operator

$$U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H't} = \mathrm{e}^{-\mathrm{i}\Delta/2}\left(\cos(\widetilde{\Omega}t/2)\mathbb{1} + \mathrm{i}\frac{\widetilde{\boldsymbol{\Omega}}\cdot\boldsymbol{\sigma}}{\widetilde{\Omega}}\sin(\widetilde{\Omega}t/2)\right)\,.$$

Thus we have

$$\rho_{ee}(t) = |\frac{\mathrm{i}(\Omega' - \mathrm{i}\Omega'')}{\widetilde{\Omega}}\sin(\widetilde{\Omega}t/2)|^2 = \frac{|\Omega|^2}{2\widetilde{\Omega}^2}\left(1 - \cos(\widetilde{\Omega}t)\right)\,.$$

The excited state population is illustrated in Figure S5.1.

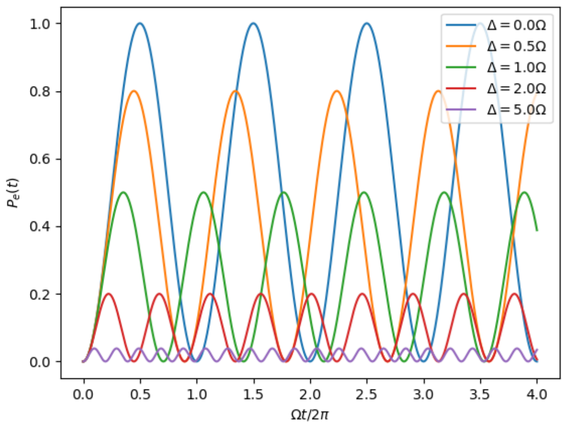

Figure S5.1: Excited state occupation $\rho_{ee}(t) = |\langle e|\, U(t)\, |\psi_0\rangle|^2$ for the initial state $|\psi_0\rangle = |g\rangle$ scanned for $\Delta = 0, \Omega/2, \Omega, 2\Omega, 5\Omega$.

S5.2 Optical Lattices

(a) Using the ansatz $|\psi\rangle = \psi_e\,|e\rangle + \psi_g\,|g\rangle$ we get

$$\begin{aligned}\mathrm{i}\hbar\partial_t\psi_e &= \frac{p^2}{2m}\psi_e - \hbar\Delta\psi_e + \frac{\hbar}{2}\Omega^*\psi_g \\ \mathrm{i}\hbar\partial_t\psi_g &= \frac{p^2}{2m}\psi_g + \frac{\hbar}{2}\Omega\psi_e\end{aligned} \tag{S5.2}$$

(b) Applying the adiabatic approximation to the first equation in (S5.2) yields

$$\psi_e = \frac{\Omega^*}{2\Delta}\psi_g\,,$$

and inserting this into the second equation, yields

$$\mathrm{i}\hbar\partial_t\psi_g = \frac{p^2}{2m}\psi_g + \frac{\hbar|\Omega|^2}{4\Delta}\psi_g\,,$$

which immediately yields the effective potential V_{eff}.

(c) We have $\Omega(x) = \Omega_0 \cos(kx)$.

(i) By computing a quadratic approximation at a minimum of the effective potential we get an approximation of

$$V_{\text{eff}}(x) = \frac{\hbar\Omega_0^2}{4\Delta}\cos(kx)^2 = \frac{\hbar\Omega_0^2}{8\Delta}(1+\cos(2kx)) \approx \frac{1}{2}m\frac{\hbar\Omega_0^2 k^2}{2m\Delta}(x-\pi/2k)^2 .$$

The height of the potential is obviously given by

$$V_{\text{eff}}^{\max} = \frac{\hbar\Omega_0^2}{4\Delta},$$

the frequency of the quadratic expansion is

$$\omega_Q^2 = \frac{\hbar\Omega_0^2 k^2}{2m\Delta},$$

and the energies of the quadratic potential are given by

$$E_Q(N) = \hbar\omega_Q(N+\frac{1}{2}).$$

Thus we can estimate

$$\begin{aligned} E_Q(N) &< V_{\text{eff}}^{\max} \\ \Leftrightarrow \hbar^2\frac{\hbar\Omega_0^2 k^2}{2m\Delta}(N+1/2)^2 &< \frac{\hbar^2\Omega_0^4}{16\Delta^2} \\ \Leftrightarrow (N+1/2)^2 &< \frac{m\Omega_0^2}{8\hbar\Delta} \\ \Leftrightarrow N &< \sqrt{\frac{m\Omega_0^2}{8\hbar\Delta}} - 1/2 \end{aligned}$$

(ii) To obtain a trapping potential that is moving with constant velocity v we use the adiabatic approximation for the laser field

$$\begin{aligned} E(x,t) &= \frac{E_0}{2}(\cos(kx-\omega t)+\cos(kx+\phi(t))) \\ &= E_0\cos\left(\frac{k+k}{2}x+\frac{\phi(t)-\omega t}{2}\right)\cos\left(\frac{k-k}{2}x-\frac{\phi(t)+\omega t}{2}\right), \end{aligned}$$

with $\phi(t) = (\omega+\delta\omega)t$. This gives

$$E(x,t) = \frac{E_0}{2}(\cos(kx+\delta\omega t)\cos(\omega t+\delta\omega t/2)) .$$

Since $\delta\omega \ll \omega$ we can approximate

$$\Omega(x) = \Omega_0\cos(kx+\delta\omega t) .$$

If we choose $\delta\omega = \pm vk$.

(iii) By the same arguments as before using $\phi(t) = \int_0^t \omega \pm kv(t')\,\mathrm{d}t'$, yields

$$\Omega(x) = E_0\cos\left(k\left(x+\int_0^t v(t')\,\mathrm{d}t'\right)\right),$$

which is just a potential following the acceleration implicit in $v(t)$.

(iv) To estimate the maximal acceleration we can employ a classical model and calculate the maximal restoring force of the effective potential

$$F_{\text{eff}}^{\max} = \max \partial_x V_{\text{eff}}(x) = \max \frac{\hbar k \Omega_0}{4\Delta} \sin(2kx) = \frac{\hbar k \Omega_0^2}{4\Delta}.$$

So the maximal acceleration can be estimated by

$$a_{\max} = F_{\text{eff}}^{\max}/m = \frac{\hbar k \Omega_0^2}{4m\Delta}.$$

For shallow lattices a more detailed quantum mechanical analysis needs to be performed and the maximal acceleration is much smaller.

S5.3 Hyperfine Transitions for Trapped Atoms

(a) Inserting the ansatz into the time-dependent Schrödinger equation we get

$$\begin{aligned} \mathrm{i}\hbar\partial_t\psi_e &= -\hbar\Delta\psi_e + \frac{\hbar\Omega_1}{2}\psi_{g_1} + \frac{\hbar\Omega_2}{2}\psi_{g_2} \\ \mathrm{i}\hbar\partial_t\psi_{g_1} &= \hbar(\Delta_1 - \Delta)\psi_{g_1} + \frac{\hbar\Omega_1^*}{2}\psi_e \\ \mathrm{i}\hbar\partial_t\psi_{g_2} &= \hbar(\Delta_2 - \Delta)\psi_{g_2} + \frac{\hbar\Omega_2^*}{2}\psi_e \end{aligned}$$

(b) Making the adiabatic approximation we obtain

$$\psi_e \approx \frac{\Omega_1}{2\Delta}\psi_{g_1} + \frac{\Omega_2}{2\Delta}\psi_{g_2}.$$

This yields the effective two level system

$$\begin{aligned} \mathrm{i}\hbar\partial\psi_{g_1} &= \hbar(\Delta_1 - \Delta)\psi_{g_1} + \frac{\hbar|\Omega_1|^2}{2\Delta}\psi_{g_1} + \frac{\hbar\Omega_1^*\Omega_2}{2\Delta}\psi_{g_2} \\ \mathrm{i}\hbar\partial\psi_{g_2} &= \hbar(\Delta_2 - \Delta)\psi_{g_2} + \frac{\hbar|\Omega_2|^2}{2\Delta}\psi_{g_2} + \frac{\hbar\Omega_2^*\Omega_1}{2\Delta}\psi_{g_1}. \end{aligned}$$

So we get the effective two-level Hamiltonian

$$\begin{aligned} H_{\text{eff}} = &\left(\hbar(\Delta_1 - \Delta) + \frac{\hbar|\Omega_1|^2}{2\Delta}\right)|\psi_{g_1}\rangle\langle\psi_{g_1}| + \left(\hbar(\Delta_2 - \Delta) + \frac{\hbar|\Omega_2|^2}{2\Delta}\right)|\psi_{g_2}\rangle\langle\psi_{g_2}| \\ &+ \frac{\hbar\Omega_1^*\Omega_2}{2\Delta}|\psi_{g_1}\rangle\langle\psi_{g_2}| + \frac{\hbar\Omega_1\Omega_2^*}{2\Delta}|\psi_{g_2}\rangle\langle\psi_{g_1}|. \end{aligned}$$

So we get the Raman Rabi frequency

$$\Omega_{\text{R}} = \frac{\Omega_1^*\Omega_2}{\Delta}$$

and the Raman detuning with resonance condition

$$\Delta_{\text{R}} = (\Delta_1 - \Delta_2) + \frac{|\Omega_1|^2 - |\Omega_2|^2}{2\Delta} \stackrel{!}{=} 0,$$

by setting the effective energies equal.

S6 Schrödingers Cat

10 points *Markus Aichhorn*, TU Graz

(a) Recall that $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial x}$ and $\hat{x} = \mathrm{i}\hbar\frac{\partial}{\partial p}$. For $\hat{X}$ and $\hat{P}$ this yields

$$\hat{P} = \frac{1}{\sqrt{m\omega\hbar}}\hat{p} = -\mathrm{i}\frac{\partial}{\partial(\sqrt{m\omega/\hbar}\, x)} = -\mathrm{i}\frac{\partial}{\partial X}$$
$$\hat{X} = \sqrt{m\omega/\hbar}\hat{x} = \mathrm{i}\frac{\partial}{\partial(p\,/\sqrt{m\omega\hbar})} = \mathrm{i}\frac{\partial}{\partial P}\,.$$

Now we calculate the ground state, as an eigenstate of the annihilation operator, and we get

$$\langle x|\,\hat{a}\,|0\rangle = \langle x|\,\frac{1}{\sqrt{2}}(\hat{X} + \mathrm{i}\hat{P})\,|0\rangle = 0$$
$$\Leftrightarrow (X + \partial_X)\psi_0(X) = 0$$
$$\Rightarrow \psi_0(X) = \mathrm{e}^{-X^2/2}$$

and in momentum space

$$\langle p|\,\hat{a}\,|0\rangle = \langle x|\,\frac{1}{\sqrt{2}}(\hat{X} + \mathrm{i}\hat{P})\,|0\rangle = 0$$
$$\Leftrightarrow (\mathrm{i}\partial_P + \mathrm{i}P)\psi_0(P) = 0$$
$$\Rightarrow \psi_0(P) = \mathrm{e}^{-P^2/2}$$

(b) For a coherent state $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, thus we get

$$\frac{1}{\sqrt{2}}(X + \partial_X)\psi_\alpha(X) = \alpha\psi_\alpha(X)$$
$$\frac{\mathrm{i}}{\sqrt{2}}(\partial_P + P)\psi_\alpha(P) = \alpha\psi_\alpha(P)$$

yielding

$$\psi_\alpha(X) = \mathrm{e}^{-(X-\sqrt{2}\alpha)^2/2}$$
$$\psi_\alpha(P) = \mathrm{e}^{-(P+\mathrm{i}\sqrt{2}\alpha)^2/2}$$

(c) We start in the coherent state $|\alpha_0\rangle$ with $\alpha_0 = \rho\mathrm{e}^{\mathrm{i}\phi}$ at time $t = 0$. For the eigenstates $|n\rangle$ of the Hamiltonian we have the energy $E_n = \hbar\omega(n + 1/2)$. So we get

$$U(t)\,|\alpha_0\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\,|\alpha_0\rangle = \mathrm{e}^{-\rho^2/2}\sum_n \frac{\rho^n\mathrm{e}^{\mathrm{i}\phi n}}{\sqrt{n!}}\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}Ht}\,|n\rangle = \mathrm{e}^{-\rho^2/2}\sum_n \frac{\rho^n\mathrm{e}^{\mathrm{i}\phi n}}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}\omega nt}\mathrm{e}^{-\mathrm{i}\omega t/2}\,|n\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-\rho^2/2}\sum_n \frac{\rho^n\mathrm{e}^{i(\phi-\omega t)n}}{\sqrt{n!}}\,|n\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}\,|\alpha(t)\rangle\,,$$

with $\alpha(t) = \rho e^{i(\phi-\omega t)}$. For the expectation values this gives

$$\begin{aligned}\langle x\rangle_t &= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha(t)|\,\hat{a}+\hat{a}^\dagger\,|\alpha(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha(t)+\alpha(t)^*)\\ &= \sqrt{\frac{\hbar}{2m\omega}}\rho\cos(\omega t-\phi) = x_{\max}\cos(\omega t-\phi)\\ \langle p\rangle_t &= -i\sqrt{\frac{m\hbar\omega}{2}}\langle\alpha(t)|\,\hat{a}-\hat{a}^\dagger\,|\alpha(t)\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}(\alpha(t)-\alpha(t)^*)\\ &= -\sqrt{\frac{m\hbar\omega}{2}}\rho\sin(\omega t-\phi) = -p_{\max}\sin(\omega t-\phi)\,.\end{aligned}$$

Now if $|\alpha| = \rho \gg 1$ we have

$$\begin{aligned}\Delta x/\langle x\rangle_{\max} &= 1/\rho \ll 1\\ \Delta p/\langle p\rangle_{\max} &= 1/\rho \ll 1\,.\end{aligned}$$

So the uncertainty is negligible small compared to the extension of the state in position space as well as in momentum space.

(d) For the ideal pendulum we have $\omega = \sqrt{\ell/g} \approx 3.13\ \mathrm{s}^{-1}$. We have $p_0 = 0$ and thus $\phi = 0$, and $\rho = \langle x\rangle_{\max}\sqrt{\frac{m\omega}{2\hbar}} \approx 3.5\times 10^9$ and $\frac{\Delta x}{\langle x\rangle_{\max}} \approx 1.3\times 10^{-10}$.

(e) For the cat state $|\psi_c\rangle$ as in (6.4) with $\alpha = i\rho$ we get

$$\begin{aligned}\langle\alpha|x|\alpha\rangle &= 0\\ \langle\alpha|p|\alpha\rangle &= p_{\max} = \rho\sqrt{m\hbar\omega/2}\\ \langle-\alpha|x|-\alpha\rangle &= 0\\ \langle-\alpha|p|-\alpha\rangle &= -p_{\max}\,.\end{aligned}$$

If we interpret the left moving state as "dead" and the right moving as "alive" we get a superposition of two classical-like states, analogous to the Schrödinger cat.

(f) To calculate the probability density in position and momentum space we calculate

$$\begin{aligned}\mathbb{P}(X) &\propto |e^{-i\pi/4}\psi_\alpha(X) + e^{i\pi/4}\psi_{-\alpha}(X)|^2\\ &\propto |e^{-i\pi/4}e^{-(X-i\rho\sqrt{2})^2/2} + e^{i\pi/4}e^{-(X+i\rho\sqrt{2})^2/2}|^2\\ &\propto 4e^{2\rho^2-X^2}\cos\left(\sqrt{2}\rho X - \frac{\pi}{4}\right)^2\end{aligned} \tag{S6.1}$$

$$\begin{aligned}\mathbb{P}(P) &\propto |e^{-i\pi/4}\psi_\alpha(P) + e^{i\pi/4}\psi_{-\alpha}(P)|^2\\ &\propto |e^{-i\pi/4}e^{-(P-\rho\sqrt{2})^2/2} + e^{i\pi/4}e^{-(P+\rho\sqrt{2})^2/2}|^2\\ &\propto e^{-(P-\rho\sqrt{2})^2} + e^{-(P+\rho\sqrt{2})^2}\end{aligned} \tag{S6.2}$$

(g) Since the resolution is much larger than the uncertainty of the state itself and much smaller that the difference in momentum of the two states, Alice will obtain a combination of two Gaussians with width δp as distribution, one centered at $+p_0$ the other centered at $-p_0$.

(h) In momentum space the quantum superposition vanishes as seen in (f). Therefore the histogram of a classical mixture of the two states will look exactly the same as the measurement of Alice.

(i) When measuring in position space with a resolution $\delta x \ll \frac{\sqrt{\hbar/2m\omega}}{\rho} = \frac{\Delta x}{\rho}$. In this case the resolution is fine enough to resolve the structure of the quantum probability distribution given in (S6.1), and thus the distribution will have that form. On the other side, if we consider a statistical mixture of the two coherent states the probability distribution is given by

$$\begin{aligned}\mathbb{P}_{\mathrm{cl}}(X) &\propto |\psi_\alpha(X)|^2 + |\psi_{-\alpha}(X)|^2 \\ &\propto \mathrm{e}^{2\rho^2 - X^2} \,.\end{aligned} \tag{S6.3}$$

and the interference pattern does not occur. So this measurement can decide whether we only have a statistical mixture or a quantum superposition.

(j) The resolution that is necessary to see the superposition is $\delta x \ll \frac{\sqrt{\hbar/2m\omega}}{\rho} \approx 10^{-26}$, which is not achievable in practice.

S7 Boltzmann Machine

8 points *Johannes Krondorfer*, TU Graz

S7.1 Classical Boltzmann Machine

(a) For proper normalization of P we immediately get $Z = \sum_{\boldsymbol{s}} \mathrm{e}^{-\beta E(\boldsymbol{s})}$. For the information we get

$$-\log P(\boldsymbol{s}) = \beta E(\boldsymbol{s}) + \log Z = \beta\left(-\sum_{i<j} w_{ij} s_i s_j - \sum_i b_i s_i\right) + \log Z \tag{S7.1}$$

(b) This is easily shown by

$$\begin{aligned}
\langle E\rangle_P - \frac{1}{\beta}S &= \frac{1}{\beta}\sum_{\boldsymbol{s}} \left(\beta E(\boldsymbol{s})P(\boldsymbol{s}) + P(\boldsymbol{s})\log P(\boldsymbol{s})\right) \\
&= \frac{1}{\beta}\sum_{\boldsymbol{s}} \left(\left(-\log P(\boldsymbol{s}) - \log Z\right) P(\boldsymbol{s}) + P(\boldsymbol{s})\log P(\boldsymbol{s})\right) \\
&= -\frac{1}{\beta}\log Z \sum_{\boldsymbol{s}} P(\boldsymbol{s}) = -\frac{1}{\beta}\log Z = F
\end{aligned}$$

(c) We can bound the logarithm with $\log x \leqslant x - 1$ for all $x > 0$. For the Kullback-Leibler divergence this gives

$$\begin{aligned}
-D(P_{\text{data}}||P) &= \sum_{\boldsymbol{s}} P_{\text{data}}(\boldsymbol{s}) \log \frac{P(\boldsymbol{s})}{P_{\text{data}}(\boldsymbol{s})} \\
&\leqslant \sum_{\boldsymbol{s}} P_{\text{data}}(\boldsymbol{s}) \left(\frac{P(\boldsymbol{s})}{P_{\text{data}}(\boldsymbol{s})} - 1\right) = \sum_{\boldsymbol{s}} \left(P(\boldsymbol{s}) - P_{\text{data}}(\boldsymbol{s})\right) = 0\,.
\end{aligned}$$

Thus $D(P_{\text{data}}||P) \geqslant 0$. For the bound of the logarithm we have equality if and only if $x = 1$, i.e. $P_{\text{data}} = P$, which shows the statement.

(d) The Kullback-Leibler divergence can be rewritten as

$$D(P_{\text{data}}||P) = \sum_{\boldsymbol{s}} \left(P_{\text{data}}(\boldsymbol{s}) \log P_{\text{data}}(\boldsymbol{s}) - P_{\text{data}}(\boldsymbol{s}) \log P(\boldsymbol{s})\right) = -\langle \log P\rangle_{P_{\text{data}}} + \%\,,$$

and thus minimizing the KL divergence is the same as maximizing the log-likelihood $\langle \log P\rangle_{P_{\text{data}}}$.

(e) To calculate the update we first compute

$$-\frac{\partial}{\partial\theta}\log P(\boldsymbol{s}) = \beta\partial_\theta E(\boldsymbol{s}) + \frac{1}{Z}\partial_\theta Z = \beta\left(\partial_\theta E(\boldsymbol{s}) - \langle\partial_\theta E(\boldsymbol{s})\rangle_P\right)\,.$$

So we get

$$\partial_\theta D(P_{\text{data}}||P) = \beta\left(\langle\partial_\theta E\rangle_P - \langle\partial_\theta E\rangle_{P_{\text{data}}}\right)\,,$$

which means

$$\partial_{b_i} D(P_{\text{data}}||P) = \beta \left(-\langle s_i\rangle_P + \langle s_i\rangle_{P_{\text{data}}}\right)$$
$$\partial_{w_{ij}} D(P_{\text{data}}||P) = \beta \left(-\langle w_{ij}\rangle_P + \langle w_{ij}\rangle_{P_{\text{data}}}\right) .$$

Thus the update is essentially matching the expectation of model and data distribution of the computational units of the model. When the model is perfectly trained, i.e. the gradient is zero, then the expectations are the same for model and true data.

(f) In order to compute the conditional probability we use the definition

$$P(s_i = 1|\boldsymbol{s}_{-i}) = \frac{P(s_i = 1, \boldsymbol{s}_{-i})}{P(\boldsymbol{s}_{-i})} = \frac{P(s_i = 1, \boldsymbol{s}_{-i})}{P(s_i = 1, \boldsymbol{s}_{-i}) + P(s_i = -1, \boldsymbol{s}_{-i})}$$
$$= \frac{1}{1 + \frac{P(s_i=-1,\boldsymbol{s}_{-i})}{P(s_i=1,\boldsymbol{s}_{-i})}} .$$

Furthermore we have

$$\frac{P(s_i = -1, \boldsymbol{s}_{-i})}{P(s_i = 1, \boldsymbol{s}_{-i})} = \exp\left(-\beta(E(s_i = -1) - E(s_i = 1))\right) = \mathrm{e}^{-\beta\Delta E(\boldsymbol{s}_{-i})} .$$

So $\alpha = \beta$ and

$$x = \Delta E(\boldsymbol{s}_{-i}) = 2\sum_{j\neq i} w_{ij} s_j + 2b_i .$$

S7.2 Quantum Boltzmann Machine

(a) Analogously to the classical case we compute

$$\mathrm{tr}\{\rho H\} - \frac{1}{\beta} S = \frac{1}{\beta}\mathrm{tr}\left\{-\rho\log\rho - \rho\log Z + \rho\log\rho\right\} = -\frac{1}{\beta}\log Z\mathrm{tr}\{\rho\} = F$$

(b) Analogously to the classical case we perform the derivative with respect to the parameters θ_μ, yielding

$$\partial_{\theta_\mu} D(\rho_{\text{data}}||\rho) = \beta\mathrm{tr}\left\{\rho_{\text{data}}\left(\partial_{\theta_\mu} H - \langle\partial_{\theta_\mu} H\rangle_\rho\right)\right\}$$
$$= \beta\left(\langle O_\mu\rangle_{\rho_{\text{data}}} - \langle O_\mu\rangle_\rho\right) .$$

This is essentially the same as in the classical case, but with expectation values of hermitian operators in a Hilbert space instead of classical statistical expectations.

(c) The given Hamiltonian does not provide an advantage of the quantum system over the classical one, since it is diagonal in the computational basis. Thus by computing the matrix elements of the Hamiltonian we get

$$\langle \boldsymbol{s}|\, H\, |\boldsymbol{s}'\rangle = -\delta_{\boldsymbol{s}\boldsymbol{s}'}\left(\sum_{i<j} w_{ij} s_i s_j + \sum_i b_i s_i\right) ,$$

with $s_i \in \{-1, 1\}$. This is equivalent to the classical system. To exploit the advantage of the quantum system we need mixtures and superpositions of the computational basis. This could be achieved by adding terms like

$$H_I \in \left\{ \sum_i \Gamma_i \sigma_i^x \,, \quad \sum_{i<j} W_{ij} \sigma_i^x \sigma_j^x \,, \quad \sum_{ij} T_{ij} \sigma_i^x \sigma_j^z \,, \quad \dots \right\} .$$

Superpositions are incorporated since $\langle \boldsymbol{s} | \, H_I \, | \boldsymbol{s}' \rangle \not\propto \delta_{\boldsymbol{s}\boldsymbol{s}'}$

S8 Precession

14 points

Matthias Diez, TU Graz & KFU

S8.1 The Geodesic Equation

(a) The Euler Lagrange equation for $L = \sqrt{g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$ is:

$$\frac{\partial}{\partial x^\mu}L = \frac{d}{ds}\frac{\partial}{\partial \dot{x}_\mu}L \tag{S8.1}$$

where $\dot{x}_\mu = \frac{dx^\mu}{ds}$. From this we immediately get:

$$\frac{1}{2L}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{d}{ds}\left(\frac{1}{L}g_{\alpha\mu}\dot{x}^\alpha\right) \tag{S8.2}$$

Now we can use that $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$, as well as for timelike trajectories of massive particles and we thus get:

$$\frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = g_{\mu\alpha}\frac{d^2}{ds^2}x^\alpha + \frac{1}{2}\partial_\beta g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\beta + \partial_\beta g_{\alpha\mu}\dot{x}^\alpha\dot{x}^\beta \tag{S8.3}$$

Additionally we use that $g^{\mu\nu}g_{\mu\alpha}" = \delta^\nu_\alpha$ and after rearranging we arrive at:

$$\frac{d^2}{du^2}x^\mu - \Gamma^\mu_{\rho\nu}\frac{dx^\rho}{du}\frac{dx^\mu}{du} = 0, \tag{S8.4}$$

(b) We use again the Euler-Lagrange equation, this time for the Lagrangian L^2:

$$\frac{d}{ds}\left[\frac{\partial}{\partial \dot{x}_\mu}L^2\right] = \partial_\mu L^2 \tag{S8.5}$$

Using the product rule we have:

$$L\frac{d}{ds}\left[\frac{\partial}{\partial \dot{x}_\mu}L\right] = L\partial_\mu L \tag{S8.6}$$

where we used that $L = 1$ and thus $\frac{d}{ds}L = 0$.

S8.2 The Schwarzschild Metric

(a) The Lagrangian can be read off to be :

$$L = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{S8.7}$$

(b) We immediately see that L does not depend on t, ϕ and s. Therefrom we know that $\frac{\partial}{\partial t}L$, $\frac{\partial}{\partial \phi}L$ and $\frac{d}{ds}L - \dot{x}^\mu\frac{\partial L}{\partial \dot{x}^\mu}$ are constants of motion. However as we know that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$ is also a constant of motion we find

$$h = r^2\sin^2\theta\dot{\phi} \tag{S8.8}$$

$$k = \left(1 - \frac{2M}{r}\right)\dot{t} \tag{S8.9}$$

$$1 = \left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{S8.10}$$

(c) In order to get the Christoffel symbols in this specific problem we only need to evaluate the Euler Lagrange equations for our choice of coordinates and then identify the Christoffel symbols, eg.:

$$\left(1-\frac{2M}{r}\right)\frac{d^2}{ds^2}t+\frac{2M}{r^2}\dot{t}\dot{r}=0 \tag{S8.11}$$

We can rearrange this:

$$\frac{d^2}{ds^2}t+\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\dot{t}\dot{r}=0 \tag{S8.12}$$

We can read then off $\Gamma^0_{01}=\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}$.

S8.3 Classical Treatment

(a)

$$T=\frac{m}{2}\left[\dot{r}^2+r^2\dot{\phi}^2\right] \tag{S8.13}$$

(b)

$$L=\frac{m}{2}\left[\dot{r}^2+r^2\dot{\phi}^2\right]+\frac{mMG}{r} \tag{S8.14}$$

The constants of motion are:

$$h=mr^2\dot{\phi}^2 \tag{S8.15}$$

$$E=\frac{m}{2}\left[\dot{r}^2+r^2\dot{\phi}^2\right]-\frac{mMG}{r} \tag{S8.16}$$

(c) Rearranging of the equations from above gives:

$$\dot{r}^2=\frac{2(E-\frac{mMG}{r})-\frac{h^2}{mr^2}}{m} \tag{S8.17}$$

(d) First we use the chain rule to get:

$$\frac{dr}{dt}=\frac{dr}{d\phi}\frac{d\phi}{dt} \tag{S8.18}$$

Furthermore we have:

$$\frac{dp}{d\phi}=\frac{d}{d\phi}\frac{1}{r}=-\frac{1}{r^2}\frac{dr}{d\phi} \tag{S8.19}$$

Combining these two leads to:

$$\frac{dr}{dt}=-\frac{dp}{d\phi}\frac{h}{m} \tag{S8.20}$$

Inserting this in equation (S8.17) gives:

$$\left(\frac{dp}{d\phi}\right)^2+p^2=\frac{2Em}{h^2}-\frac{2MGm^2}{h^2}p \tag{S8.21}$$

Therefrom we have:

$$A=-\frac{2MGm^2}{h^2} \tag{S8.22}$$

$$B=\frac{2Em}{h^2} \tag{S8.23}$$

$$\tag{S8.24}$$

(e) Differentiating both sides with respect to ϕ gives:

$$2p'p'' + 2pp' = Ap' \tag{S8.25}$$

From this we have:

$$p'' + p = \frac{A}{2} \tag{S8.26}$$

This equation is easy to solve, the fundamental solutions are $\sin\phi$ and $\cos\phi$. We choose for the homogeneous part $\cos\phi$ and the particular solution is simply $\frac{A}{2}$. Thus we get an ellipse for $A < 0$ in polar coordinates:

$$\frac{1}{r} = p = \frac{1}{R_0}\left(\cos(\phi) + \frac{AR_0}{2}\right) \tag{S8.27}$$

where R_0 and the energy E as well as h depend on the initial conditions.

S8.4 Relativistic Treatment

(a)

$$\dot{r}^2 = k^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \frac{h^2}{r^2}\right] \tag{S8.28}$$

(b) Similar to the Newtonian case we use:

$$\frac{dr}{ds} = -\frac{dp}{d\phi}h \tag{S8.29}$$

and get:

$$p'^2 + p^2 = \frac{k^2}{h^2} - \frac{1}{h^2} + 2Mp\left[\frac{1}{h^2} + p^2\right] \tag{S8.30}$$

where $p' = \frac{dp}{d\phi}$.

(c) By differentiation of both sides we have:

$$2p'p'' + 2p'p = \frac{2M}{h^2}p' + 3Mp^2p'\,, \tag{S8.31}$$

dividing by p' leads to:

$$p'' + p = \frac{2M}{h^2} + 3Mp^2 \tag{S8.32}$$

and this gives us for C and D:

$$C = \frac{2M}{h^2} \tag{S8.33}$$

$$D = 3M \tag{S8.34}$$

(d) Justify why the term Dp^2 can be seen as a small perturbation for Mercury.

(e) Using the Newtonian solution in the perturbation we get:

$$p'' + p = \frac{2M}{h^2} + \frac{3M^3}{h^4}(1 + 2e\cos\phi + e^2\cos^2\phi) \tag{S8.35}$$

Of the particular solution the first part adds a minute constant, the third a minute constant and a periodic wiggle, but the second adds something that is not periodic with 2π and this is observable as precession. So as our approximate solution we take:

$$p = \frac{M}{h^2}(1 + e\cos\phi + \frac{3M^2}{h^2}e\phi\sin\phi) \tag{S8.36}$$

$$\approx \frac{M}{h^2}(1 + e\cos\left(\phi - \frac{3M^2}{h^2}\phi\right), \tag{S8.37}$$

where we used: $\cos(\phi - \beta) = \cos\phi\cos\beta + \sin\phi\sin\beta$ and $\cos\beta \approx 1$ and $\sin\beta \approx \beta$ for a small angle $\beta = \frac{3M^2\phi}{h^2}$. Thus the precession Δ is

$$\Delta = \frac{2\pi}{1 - \frac{3M^2}{h^2}} - 2\pi \approx 6\pi\frac{M^2}{h^2}. \tag{S8.38}$$

S9 Pulsar Electrodynamics

10 points *Martin Napetschnig*, TU Munich

S9.1 Pulsar Characteristics

(a)

$$P_{\rm min} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{S9.1}$$

I get for the Sun $P_{\rm min} = 1.03 \times 10^4\,$s while I get for the NS with the above values $P_{\rm min} = 5.85 \times 10^{-4}\,$s. The answer is that the Sun is two orders of magnitude below above their minimum period, while pulsars are really close to it. The result giving values below ms can be explained by order 1 factors in the masses and radii of pulsars.

(b) **(0.5 points)** $\frac{\Omega_f}{\Omega_i} = \frac{B_f}{B_i} = \left(\frac{R_i}{R_f}\right)^2$. Taking the indicated values gives $\frac{B_f}{B_i} = \frac{R_\odot^2}{R_{NS}^2} \sim 4.9 \times 10^5$ as a magnification factor for the magnetic field, which is the same as for the angular momentum.

(c)

$$\vec{\mu} = |\vec{\mu}| \begin{pmatrix} \sin\alpha\ \cos(\Omega t) \\ \sin\alpha\ \sin(\Omega t) \\ \cos\alpha \end{pmatrix}, \tag{S9.2}$$

$$\frac{d}{dt}E_{\rm kin} = \frac{d}{dt}\left(\frac{I\Omega^2}{2}\right), \tag{S9.3}$$

$$I\Omega\dot{\Omega} = \frac{2}{5}MR^2\Omega\dot{\Omega} = -\frac{\mu_0}{12\pi c^3}\mu^2\sin^2\alpha\,\Omega^4, \tag{S9.4}$$

$$\dot{\Omega} = -\left(\frac{\mu_0\mu^2\sin^2\alpha}{12\pi c^3 I}\right)\Omega^3, \tag{S9.5}$$

$$\dot{P} = \left(\frac{\mu_0\mu^2\pi\sin^2\alpha}{3c^3 I}\right)P^{-1}. \tag{S9.6}$$

(d) We integrate (S9.6):

$$\int_{P_{\rm initial}}^{P_0} dP\,P = \left(\frac{\mu_0\mu^2\pi\sin^2\alpha}{3c^3 I}\right)\int_0^T dt, \tag{S9.7}$$

$$\int_{P_{\rm initial}}^{P_0} dP\,P = P_0\,\dot{P}_0\int_0^T dt, \tag{S9.8}$$

$$\frac{P_0^2}{2}\left(1 - \frac{P_{\rm initial}^2}{P_0^2}\right) = P_0\,\dot{P}_0\,T \rightarrow \tag{S9.9}$$

$$T \sim \frac{1}{2}\frac{P_0}{\dot{P}_0} \tag{S9.10}$$

S9.2 The Aligned Rotator

(a) Using (9.7) and the index notation:

$$\partial_i \partial_j \frac{\mu_j}{(r_k r_k)^{\frac{1}{2}}} = \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{\mu}}{r}\right) = \tag{S9.11}$$

$$\partial_i \left(\frac{\mu_j r_j}{(r_k r_k)^{\frac{3}{2}}}\right) = \vec{\nabla}\frac{(\vec{\mu}\cdot\vec{r})}{r^3} = \tag{S9.12}$$

$$\frac{\mu_i}{r^3} - 3\frac{\mu_j r_j r_i}{r^5}, \Rightarrow \tag{S9.13}$$

$$\vec{B}_{out} = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\, r^2 - 3(\vec{\mu}\cdot\vec{r})\,\vec{r}}{r^5} = \tag{S9.14}$$

$$= -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}\left(2\cos\theta\,\vec{e}_r + \sin\theta\,\vec{e}_\theta\right) \tag{S9.15}$$

In the last line I used (9.6) for $\vec{e}_z$. The magnetic moment at the pole is given by:

$$\mu = \frac{2\pi B_0 R^3}{\mu_0}. \tag{S9.16}$$

(b) We have

$$\int \frac{dr}{r} = \int d\theta\, \frac{B_r}{B_\theta} \tag{S9.17}$$

$$= 2\int d\theta\, \frac{\cos\theta}{\sin\theta} \Rightarrow \tag{S9.18}$$

$$r(\theta) = K\sin^2\theta \tag{S9.19}$$

(c) There are two ways to derive this. Ones uses (9.11), the other uses (9.9). I will show first the latter option here and use also $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{B} = \mu_0\vec{j}_{out} - \frac{\partial}{\partial t}\vec{E} = 0$:

$$0 = \vec{E} + (\vec{\Omega}\times\vec{r})\times\vec{B}, \tag{S9.20}$$

$$\vec{E} = -(\vec{\Omega}\cdot\vec{B})\vec{r} + \vec{\Omega}(\vec{B}\cdot\vec{r}), \tag{S9.21}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{GJ}}{\varepsilon_0} = -\partial_i\left[r_i\Omega_j B_j - B_j\Omega_i r_j\right], \tag{S9.22}$$

$$\rho_{GJ} = -\varepsilon_0\left[3\Omega_j B_j + \Omega_j r_i \partial_i B_j - \Omega_i r_j \partial_i B_j - B_i\Omega_i\right] \tag{S9.23}$$

$$= -\varepsilon_0\left[2(\vec{\Omega}\cdot\vec{B}) + \frac{1}{6}\varepsilon_{ijk}\varepsilon^{ijk}\Omega_i r_j(\partial_i B_j - \partial_j B_i)\right] \tag{S9.24}$$

$$= -2\varepsilon_0(\vec{\Omega}\cdot\vec{B}) - \frac{\varepsilon_0}{6}(\vec{\Omega}\times\vec{r})(\vec{\nabla}\times\vec{B}), \Rightarrow \tag{S9.25}$$

$$\rho_{GJ} = -2\varepsilon_0(\vec{\Omega}\cdot\vec{B}) \tag{S9.26}$$

The other way is smoother (using also (9.10)):

$$0 = \vec{E} + (\vec{\Omega}\times\vec{r})\times\vec{B}, \tag{S9.27}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{GJ}}{\varepsilon_0} = -\vec{\nabla}\cdot\left[(\vec{\Omega}\times\vec{r})\times\vec{B}\right] = \tag{S9.28}$$

$$= -\vec{B}\cdot(\vec{\nabla}\times(\vec{\Omega}\times\vec{r})) = \tag{S9.29}$$

$$= -\vec{B}\cdot(3\vec{\Omega} - (\vec{\Omega}\cdot\vec{\nabla})\vec{r}), \Rightarrow \tag{S9.30}$$

$$\rho_{GJ} = -2\varepsilon_0(\vec{\Omega}\cdot\vec{B}) \tag{S9.31}$$

After carrying out the scalar product (giving 0.5 P for somebody who does this directly without computing ρ_{GJ}):

$$\rho_{GJ} = +\text{const}\left(2\cos^2\theta - \sin^2\theta\right) \sim \left(3\cos^2\theta - 1\right) \tag{S9.32}$$

Thus the critical angle at which there are no charges is given by $\theta_c = \cos^{-1}(\frac{1}{\sqrt{3}}) \sim 55^\circ \sim 0.95\,\text{rad}$. We have positive (negative) charges for angles below (above) this value. Of course, we have 4 quadrants: positive charges for $\theta \in [-55^\circ, +55^\circ]$ and $\theta \in [125^\circ, +235^\circ$ and negative ones otherwise. Figure 9.2 is unfortunately confusing, because these papers consider a pulsar spinning in the opposite direction to our case.

(d) The speed limit is $\Omega \cdot R_L \leqslant c \to R_L = \frac{c}{\Omega}$. Now I use (S9.19)

$$\frac{R}{R_L} = \frac{\sin^2\theta_{PC}}{\sin^2\frac{\pi}{2}}, \tag{S9.33}$$

$$\theta_{PC} \sim \sqrt{\frac{\Omega R}{c}} \tag{S9.34}$$

The small polar cap region sources open field lines.

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